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HOMOGENIZATION OF SYSTEMS WITH EQUI-INTEGRABLE COEFFICIENTS

Marc Briane[1] and Juan Casado-Díaz[2]

Abstract. In this paper we prove a H-convergence type result for the homogenization of systems the coefficients of which satisfy a functional ellipticity condition and a strong equi-integrability condition. The equi-integrability assumption allows us to control the fact that the coefficients are not equi-bounded. Since the truncation principle used for scalar equations does not hold for vector-valued systems, we present an alternative approach based on an approximation result by Lipschitz functions due to Acerbi and Fusco combined with a Meyers L^p-estimate adapted to the functional ellipticity condition. The present framework includes in particular the elasticity case and the reinforcement by stiff thin fibers.

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1. Introduction

This paper is devoted to the asymptotic behavior of vector-valued systems with M equations, in a regular bounded open set Ω of $\mathbb{R}^N$,

$$\begin{cases} -\operatorname{Div}(\mathbb{A}_n Du_n) = f & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\mathbb{A}_n$ is a sequence of equi-coercive (in a functional sense, see Eq. (2.1)) but not necessarily equi-bounded tensor-valued functions. The equi-bounded case was studied by Spagnolo [30] by G-convergence, and by Murat, Tartar [28] (see also [31]) by H-convergence. Then, in the scalar case assuming the L^1-boundedness and the equi-integrability of the sequence $|\mathbb{A}_n|$, Carbone and Sbordone proved a compactness result for the equation (1.1) using De Giorgi's Γ-convergence [20, 21] (see, also [7, 19] for a presentation of Γ-convergence). On the other hand, Fenchenko and Khruslov [23] (see also [24,25]) were the first to show the appearance of nonlocal effects in the homogenization of equation (1.1) when $|\mathbb{A}_n|$ is bounded in $L^1(\Omega)$ but not equi-integrable. To this end they considered a medium reinforced by very thin fibers. Several works [3, 8, 9, 12, 13, 15, 16, 27] have extended this seminal article on the limit closure of equations (1.1) in connection with the Beurling–Deny [6] representation of Dirichlet forms. All these contributions are strongly based on the truncation principle (which is also called

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the Markov property in Dirichlet forms theory) and the maximum principle as a by-product, which are specific to the scalar case.

The vector-valued case for which (1.1) is actually a system is much more delicate since the truncation principle does not hold. In the case of elasticity [4] the homogenization of fiber reinforced media can lead to nonlocal effects as in the scalar case. Under this particular geometry and the boundedness of $|\mathbb{A}_n|$ in $L^1(\Omega)$, the appearance of nonlocal effects is induced by the loss of equi-integrability for $|\mathbb{A}_n|$. However, contrary to the scalar case which is constrained by the Beurling-Deny representation, when the fibers are very stiff – *i.e.*, $|\mathbb{A}_n|$ is not equi-bounded in $L^1(\Omega)$ – fourth-order derivatives may appear at the limit in dimension three [29] as well as in dimension two [10, 11]. Actually, Camar-Eddine and Seppecher [17] proved that in dimension three the Γ-convergence closure for the L^2-strong topology of the elastic energies associated with equations (1.1) agrees with the whole set of the lower semi-continuity quadratic functionals which are null for the rigid displacements. This closure set thus contains functionals with nonlocal terms and derivatives at any order. In the more general framework of systems (1.1) the closure set is far to be clear. A first step in the understanding of the homogenization of such a system, with an equi-coercive but not equi-bounded sequence of tensor-valued functions, would be to know if the sole equi-integrability of $|\mathbb{A}_n|$ in $L^1(\Omega)$ implies a compactness result for the sequence (1.1) as in the scalar case [18]. Up to our knowledge there is no general result in this direction.

In this paper we prove a H-convergence type result (see Thm. 2.3) for system (1.1) assuming that there exists a functionally equi-coercive and equi-bounded sequence $\mathbb{B}_n$ of tensor-valued functions such that

$$\lim_{n\to\infty} \|\mathbb{A}_n - \mathbb{B}_n\|_{L^1(\Omega)^{(M\times N)^2}} = 0. \tag{1.2}$$

This assumption includes the case where $\mathbb{A}_n$ takes non-uniformly bounded values only in some set F_n with $|F_n| \to 0$, as in the fiber reinforcement setting (see Rem. 2.2). Contrary to the truncation of the solution u_n of (1.1) used in the scalar case [18], our method is based on the approximation result by Lipschitz functions due to Acerbi and Fusco [1, 2], which can be regarded as a truncation of the gradient Du_n. Moreover, the L^p-Meyers estimate is an alternative key ingredient of our approach. At this level we give an extension of the proof of [5] (see Prop. 3.1), which takes into account that the strong ellipticity of $\mathbb{A}_n$ is functional rather than pointwise. This functional ellipticity allows us to include the elasticity case in a general framework. Indeed, the pointwise ellipticity with respect to the symmetrized gradient combined with Korn's inequality implies a functional ellipticity with respect to the whole gradient.

In view of the compactness result [18] which is restricted to the scalar case, our conjecture is that a H-convergence result holds for system (1.1) if $\mathbb{A}_n$ is functionally equi-coercive but simply bounded and equi-integrable in $L^1(\Omega)^{(M\times N)^2}$. This is of course a weaker condition than (1.2). Very recently we have proved the conjecture in [14] but only for dimension $N = 2$. The method based on a div-curl lemma also provides a compactness result in dimension $N > 2$, under the assumption that $|\mathbb{A}_n|$ is bounded in $L^\rho(\Omega)$ with $\rho > \frac{N-1}{2}$. This condition is more restrictive than the equi-integrability in $L^1(\Omega)$, and cannot be compared to condition (1.2). In particular, it is not sharp in the fiber reinforcement setting contrary to condition (1.2). To conclude, the present approach is quite different, does work in any dimension and is well adapted to the fiber reinforcement problem.

Notations

- M and N are two positive integers.
- $\mathbb{A}$ denotes a tensor-valued function taking its values in $\mathbb{R}^{(M\times N)^2}$.
- : denotes the scalar product in $\mathbb{R}^{M\times N}$, *i.e.* $\xi : \eta = \mathrm{tr}\left(\xi^T \eta\right)$ for any $\xi, \eta \in \mathbb{R}^{M\times N}$.
- ∇u denotes the gradient of the scalar distribution $u : \mathbb{R}^N \to \mathbb{R}$.
- Du denotes the Jacobian matrix of the vector-valued distribution $u : \mathbb{R}^N \to \mathbb{R}^M$, *i.e.*

$$Du := \left[\frac{\partial u_i}{\partial x_j}\right]_{1\le i\le M,\, 1\le j\le N}.$$

- div denotes the classical divergence operator acting on the vector-valued distributions.
- Div denotes the vector-valued differential operator taking the divergence of each row of a matrix-valued distribution,

$$\mathrm{Div}\, U := \left[\sum_{j=1}^{N} \frac{\partial U_{ij}}{\partial x_j}\right]_{1\le i\le M}, \quad \text{for } U : \mathbb{R}^N \to \mathbb{R}^{M\times N}.$$

- c denotes a positive constant which may vary from line to line.

2. The main result

Let Ω be a regular bounded domain of $\mathbb{R}^N$, $N \ge 2$, and let M be a positive integer. Consider a sequence $\mathbb{A}_n$, $n \in \mathbb{N}$, of tensor-valued functions in $L^\infty(\Omega)^{(M\times N)^2}$ which satisfies the following properties:

- there exists a sequence $\mathbb{B}_n$ of tensor-valued functions in $L^\infty(\Omega)^{(M\times N)^2}$ such that

$$\forall\, u \in H^1_0(\Omega)^M, \quad \begin{cases} \alpha \displaystyle\int_\Omega |Du|^2\,\mathrm{d}x \le \min\left(\int_\Omega \mathbb{A}_n Du : Du\,\mathrm{d}x, \int_\Omega \mathbb{B}_n Du : Du\,\mathrm{d}x\right) \\ \beta^{-1} \displaystyle\int_\Omega |\mathbb{B}_n Du|^2 \le \int_\Omega \mathbb{B}_n Du : Du\,\mathrm{d}x, \end{cases} \tag{2.1}$$

  for given constants $\alpha, \beta > 0$, and

$$\lim_{n\to\infty} \|\mathbb{A}_n - \mathbb{B}_n\|_{L^1(\Omega)^{(M\times N)^2}} = 0; \tag{2.2}$$

- there exists a constant $C > 0$ such that the generalized Cauchy–Schwarz inequality holds

$$(\mathbb{A}_n\xi : \eta)^2 \le C\,(\mathbb{A}_n\xi : \xi)\,(\mathbb{A}_n\eta : \eta), \quad \text{a.e. in } \Omega,\ \forall\,(\xi,\eta) \in \mathbb{R}^{M\times N} \times \mathbb{R}^{M\times N}. \tag{2.3}$$

Remark 2.1. Assumption (2.1) implies the pointwise estimates

$$\alpha\,|\xi|^2|\eta|^2 \le \mathbb{B}_n(\xi\otimes\eta) : (\xi\otimes\eta) \le \beta\,|\xi|^2|\eta|^2, \quad \text{a.e. in } \Omega,\ \forall\,(\xi,\eta) \in \mathbb{R}^M \times \mathbb{R}^N. \tag{2.4}$$

We refer to Lemma 22.5 of [19] for a similar computation. For the reader's convenience, let us check briefly (2.4):

Putting in the first inequality of (2.1) the functions $u(x) := \varphi(x)\cos(k\,\eta\cdot x)\,\xi$, with $k \ge 1$, $\varphi \in C^1_c(\Omega)$, $(\xi,\eta) \in \mathbb{R}^M \times \mathbb{R}^N$, it follows that

$$\alpha \int_\Omega |\xi|^2|\eta|^2\,\varphi^2(x)\,\sin^2(k\,\eta\cdot x)\,\mathrm{d}x \le \int_\Omega \mathbb{B}_n(x)(\xi\otimes\eta) : (\xi\otimes\eta)\,\varphi^2(x)\,\sin^2(k\,\eta\cdot x)\,\mathrm{d}x + O(k^{-1}).$$

Then, passing to the limit as $k \to \infty$ and using the arbitrariness of φ we get the first inequality of (2.4). Similarly, we deduce from the second inequality of (2.1) that

$$\big|\mathbb{B}_n(x)(\xi\otimes\eta)\big|^2 \le \beta\,\mathbb{B}_n(x)(\xi\otimes\eta) : (\xi\otimes\eta) \quad \text{a.e. } x \in \Omega,$$

which implies

$$\big|\mathbb{B}_n(x)(\xi\otimes\eta)\big| \le \beta\,|\xi||\eta| \quad \text{a.e. } x \in \Omega, \tag{2.5}$$

and thus the second inequality of (2.4).

Now, decomposing any matrix $Q \in \mathbb{R}^{M\times N}$ on the canonical basis, namely

$$Q = \sum_{i=1}^{M}\sum_{j=1}^{N} Q_{ij}\, f_i \otimes e_j,$$

we easily obtain from inequality (2.5) the existence of a constant $\gamma > 0$ depending on β, M, N, such that

$$\big|\mathbb{B}_n(x)\big| := \max_{Q\in\mathbb{R}^{M\times N},\,|Q|=1} \big|\mathbb{B}_n(x)Q\big| \le \gamma \quad \text{a.e. } x \in \Omega. \tag{2.6}$$

Remark 2.2. Condition (2.2) is equivalent to the existence of a functionally equi-coercive, equi-bounded sequence $\tilde{\mathbb{B}}_n$ in $L^\infty(\Omega)^{(M\times N)^2}$ such that for any $\varepsilon > 0$, there exists a sequence of measurable sets F_n^ε of Ω satisfying

$$|\mathbb{A}_n - \tilde{\mathbb{B}}_n| \le \varepsilon \text{ a.e. in } \Omega \setminus F_n^\varepsilon, \quad \lim_{n\to\infty} |F_n^\varepsilon| = 0 \quad \text{and} \quad \lim_{n\to\infty} \int_{F_n^\varepsilon} |\mathbb{A}_n| \,\mathrm{d}x = 0. \tag{2.7}$$

Indeed, condition (2.7) implies convergence (2.2) with $\mathbb{B}_n := \tilde{\mathbb{B}}_n$, since

$$\limsup_{n\to\infty} \left(\int_\Omega |\mathbb{A}_n - \mathbb{B}_n| \,\mathrm{d}x \right) \le \limsup_{n\to\infty} \left(\varepsilon\,|\Omega \setminus F_n^\varepsilon| + \int_{F_n^\varepsilon} |\mathbb{A}_n| \,\mathrm{d}x + \int_{F_n^\varepsilon} |\mathbb{B}_n| \,\mathrm{d}x \right) = \varepsilon\,|\Omega|.$$

Conversely, assume that convergence (2.2) holds. For $\varepsilon > 0$, define the measurable set F_n^ε and the tensor-valued function $\tilde{\mathbb{B}}_n$ by

$$F_n^\varepsilon := \big\{ |\mathbb{A}_n - \mathbb{B}_n| > \varepsilon \big\} \quad \text{and} \quad \tilde{\mathbb{B}}_n := \mathbb{B}_n. \tag{2.8}$$

The first assertion of (2.7) is clearly satisfied. Then, by the strong convergence (2.2) Lebesgue's measure of F_n^ε tends to 0, and

$$\int_{F_n^\varepsilon} |\mathbb{A}_n| \,\mathrm{d}x \le \int_{F_n^\varepsilon} |\mathbb{A}_n - \mathbb{B}_n| \,\mathrm{d}x + |F_n^\varepsilon|\, \|B_n\|_{L^\infty(\Omega)^{(M\times N)^2}} \to 0,$$

which yields (2.7). Note that if the equi-coerciveness of $\mathbb{A}_n$ and $\mathbb{B}_n$ are pointwise rather than functional, then we may take $\varepsilon = 0$ in condition (2.7) and choose

$$F_n := \big\{ |\mathbb{A}_n - \mathbb{B}_n| > 1 \big\} \quad \text{and} \quad \tilde{\mathbb{B}}_n := \begin{cases} \mathbb{A}_n \text{ in } \Omega \setminus F_n \\ \mathbb{B}_n \text{ in } F_n. \end{cases}$$

Condition (2.7) is relevant in the case of a fiber reinforced medium for which the set F_n^ε is composed of stiff thin fibers, while the surrounding medium has uniformly bounded coefficients. In this setting, the loss of equi-integrability with respect to condition (2.7) corresponds to

$$\limsup_{n\to\infty} \int_{F_n^\varepsilon} |\mathbb{A}_n| \,\mathrm{d}x > 0. \tag{2.9}$$

In fact, condition (2.9) may lead to pathologies in the homogenization process of system (2.10): nonlocal effects in conductivity [3,9,16,23] and in elasticity [4], but also the appearance of second gradients in elasticity [10,11,29].

We have the following H-convergence type result:

Theorem 2.3. *Assume that the conditions (2.1) and (2.2) are fulfilled. Then, there exist a subsequence of n, still denoted by n, and a tensor-valued function $\mathbb{B}$ in $L^\infty(\Omega)^{(M\times N)^2}$ satisfying the estimates (2.1) such that for any distribution $f \in H^{-1}(\Omega)^M$, the solution u_n in $H_0^1(\Omega)^M$ of the equation*

$$-\operatorname{Div}(\mathbb{A}_n Du_n) = f \quad \text{in } \Omega, \tag{2.10}$$

satisfies the convergences

$$u_n \rightharpoonup u \text{ weakly in } H_0^1(\Omega)^M \quad \text{and} \quad \mathbb{A}_n Du_n \rightharpoonup \mathbb{B}Du \text{ weakly in } L^1(\Omega)^{M\times N}, \tag{2.11}$$

where u is the solution in $H_0^1(\Omega)^M$ of

$$-\operatorname{Div}(\mathbb{B}Du) = f \quad \text{in } \Omega. \tag{2.12}$$

Moreover, the limit tensor $\mathbb{B}$ only depends on the sequence $\mathbb{B}_n$.

3. Proofs

Proof of Theorem 2.3. First note that using a density argument we are led to the case where the right-hand side f of equation (2.10) belongs to $W^{-1,p}(\Omega)$ where p is the Meyers exponent obtained in Proposition 3.1 below. Due to the functional ellipticity (2.1) the sequence u_n is bounded in $H^1_0(\Omega)^M$, thus up to a subsequence converges weakly to some function u in $H^1_0(\Omega)^M$.

The Murat–Tartar H-convergence [28, 31] for linear scalar operators can be extended without restriction to the linear operators $\mathrm{Div}\,(\mathbb{B}_n D\cdot)$. Hence, there exists a subsequence of n, still denoted by n, and a tensor-valued function $\mathbb{B}$ such that for any distribution $f \in H^{-1}(\Omega)^M$, the solution $v_n \in H^1_0(\Omega)^M$ of the equation

$$-\,\mathrm{Div}\,(\mathbb{B}_n Dv_n) = f \quad \text{in } \Omega, \tag{3.1}$$

satisfies the convergences

$$v_n \rightharpoonup v \ \text{ weakly in } H^1_0(\Omega)^M \quad \text{and} \quad \mathbb{B}_n Dv_n \rightharpoonup \mathbb{B}Dv \ \text{ weakly in } L^2(\Omega)^{M\times N}, \tag{3.2}$$

where v is the solution in $H^1_0(\Omega)^M$ of (2.12). From the convergences (3.2) and the lower semi-continuity of the L^2-norm we easily deduce that the homogenized tensor $\mathbb{B}$ satisfies the estimates (2.1) and (2.6). Therefore, up to extract a subsequence we can assume that the sequence $\mathbb{B}_n$ H-converges to some tensor-valued function $\mathbb{B}$ satisfying the estimates (2.1) and (2.6).

The proof is divided in two steps:

First step: Let u_n be the solution of equation (2.10), and let $\bar{u}_n$ be the solution in $H^1_0(\Omega)^M$ of

$$-\,\mathrm{Div}\,(\mathbb{B}_n D\bar{u}_n) = f \quad \text{in } \Omega. \tag{3.3}$$

First of all, by virtue of Proposition 3.1 the sequence $\bar{u}_n$ is bounded in $W^{1,p}(\Omega)^M$. Then, by the Lusin type approximation theorem due to Acerbi and Fusco [1] (main theorem p. 1), [2] (Lem. [II-6]), for any $k \geq 1$, there exists a function $\bar{u}^k_n$ in $W^{1,\infty}(\Omega)^M \cap H^1_0(\Omega)^M$ such that

$$\|\bar{u}^k_n\|_{W^{1,\infty}(\Omega)^M} \leq k, \tag{3.4}$$

$$\left|\{\bar{u}^k_n \neq \bar{u}_n\}\right| \leq \frac{c}{k^p}\,\|\bar{u}_n\|^p_{W^{1,p}(\Omega)^M}. \tag{3.5}$$

Now, we will prove that

$$\lim_{k\to\infty}\left[\limsup_{n\to\infty}\left(\int_\Omega \mathbb{A}_n\left(Du_n - D\bar{u}^k_n\right) : \left(Du_n - D\bar{u}^k_n\right)\mathrm{d}x\right)\right] = 0. \tag{3.6}$$

By equations (2.10) and (3.3) we have

$$\begin{aligned}
&\int_\Omega \mathbb{A}_n\left(Du_n - D\bar{u}^k_n\right) : \left(Du_n - D\bar{u}^k_n\right)\mathrm{d}x \\
&= \left\langle f, u_n - \bar{u}^k_n\right\rangle - \int_\Omega \mathbb{B}_n D\bar{u}^k_n : \left(Du_n - D\bar{u}^k_n\right)\mathrm{d}x - \int_\Omega \left(\mathbb{A}_n - \mathbb{B}_n\right) D\bar{u}^k_n : \left(Du_n - D\bar{u}^k_n\right)\mathrm{d}x \\
&= \int_\Omega \mathbb{B}_n\left(D\bar{u}_n - D\bar{u}^k_n\right) : \left(Du_n - D\bar{u}^k_n\right)\mathrm{d}x - \int_\Omega \left(\mathbb{A}_n - \mathbb{B}_n\right) D\bar{u}^k_n : \left(Du_n - D\bar{u}^k_n\right)\mathrm{d}x.
\end{aligned} \tag{3.7}$$

To estimate the first term on the right-hand side of (3.7), we use that (3.4) and (3.5) imply that $\bar{u}_n^k$ is bounded in $W^{1,p}(\Omega)^M$ independently of n and k. Therefore, using the bound (2.6) and Hölder's inequality, we get

$$\begin{aligned}
&\left| \int_\Omega \mathbb{B}_n \left(D\bar{u}_n - D\bar{u}_n^k\right) : \left(Du_n - D\bar{u}_n^k\right) \mathrm{d}x \right| \\
&\leq \gamma \int_{\{\bar{u}_n^k \neq \bar{u}_n\}} \left(|D\bar{u}_n| + |D\bar{u}_n^k|\right) \left(|Du_n| + |D\bar{u}_n^k|\right) \mathrm{d}x \\
&\leq \gamma \left(\|\bar{u}_n\|_{W^{1,p}(\Omega)^M} + \|\bar{u}_n^k\|_{W^{1,p}(\Omega)^M}\right) \left(\|u_n\|_{H^1(\Omega)^M} + \|\bar{u}_n^k\|_{H^1(\Omega)^M}\right) \left|\{\bar{u}_n^k \neq \bar{u}_n\}\right|^{\frac{1}{2}-\frac{1}{p}} \\
&\leq c\, k^{1-\frac{p}{2}}.
\end{aligned} \tag{3.8}$$

To estimate the second term on the right-hand side of (3.7), we take $m > 0$ and we use the decomposition

$$\begin{aligned}
&\left| \int_\Omega (\mathbb{A}_n - \mathbb{B}_n)\, D\bar{u}_n^k : \left(Du_n - D\bar{u}_n^k\right) \mathrm{d}x \right| \\
&\leq \left| \int_{\{|Du_n| \leq m\}} (\mathbb{A}_n - \mathbb{B}_n)\, D\bar{u}_n^k : \left(Du_n - D\bar{u}_n^k\right) \mathrm{d}x \right| \\
&+ \left| \int_{\{|Du_n| > m\}} \mathbb{A}_n\, D\bar{u}_n^k : \left(Du_n - D\bar{u}_n^k\right) \mathrm{d}x \right| + \left| \int_{\{|Du_n| > m\}} \mathbb{B}_n\, D\bar{u}_n^k : \left(Du_n - D\bar{u}_n^k\right) \mathrm{d}x \right|.
\end{aligned} \tag{3.9}$$

The first term on the right-hand side of this inequality can be estimated by

$$\left| \int_{\{|Du_n| \leq m\}} (\mathbb{A}_n - \mathbb{B}_n)\, D\bar{u}_n^k : \left(Du_n - D\bar{u}_n^k\right) \mathrm{d}x \right| \leq k\,(m+k)\, \|\mathbb{A}_n - \mathbb{B}_n\|_{L^1(\Omega)^{(M\times N)^2}}. \tag{3.10}$$

Taking into account (2.3), we can estimate the second term on the right-hand side of (3.9) by

$$\begin{aligned}
&\left| \int_{\{|Du_n| > m\}} \mathbb{A}_n\, D\bar{u}_n^k : \left(Du_n - D\bar{u}_n^k\right) \mathrm{d}x \right| \\
&\leq C \left(\int_{\{|Du_n| > m\}} \mathbb{A}_n\, D\bar{u}_n^k : D\bar{u}_n^k \,\mathrm{d}x \right)^{\frac{1}{2}} \left(\int_{\{|Du_n| > m\}} \mathbb{A}_n\, Du_n : Du_n \,\mathrm{d}x \right)^{\frac{1}{2}} \\
&+ \int_{\{|Du_n| > m\}} \mathbb{A}_n\, D\bar{u}_n^k : D\bar{u}_n^k \,\mathrm{d}x \\
&\leq c\, k \left(\int_{\{|Du_n| > m\}} |\mathbb{A}_n| \,\mathrm{d}x \right)^{\frac{1}{2}} + k^2 \int_{\{|Du_n| > m\}} |\mathbb{A}_n| \,\mathrm{d}x.
\end{aligned} \tag{3.11}$$

For the third term on the right-hand side of (3.9) we use

$$\left| \int_{\{|Du_n| > m\}} \mathbb{B}_n\, D\bar{u}_n^k : \left(Du_n - D\bar{u}_n^k\right) \mathrm{d}x \right| \leq \gamma\, k\, \|u_n\|_{H^1(\Omega)^M} \left|\{|Du_n| > m\}\right|^{\frac{1}{2}} + \gamma\, k^2 \left|\{|Du_n| > m\}\right|. \tag{3.12}$$

The estimates (3.9) to (3.12) show that

$$\left| \int_\Omega (\mathbb{A}_n - \mathbb{B}_n)\, D\bar{u}_n^k : \big(Du_n - D\bar{u}_n^k\big)\, \mathrm{d}x \right| \le k\,(m+k)\, \|\mathbb{A}_n - \mathbb{B}_n\|_{L^1(\Omega)^{(M\times N)^2}}$$
$$+\, c\,k \left(\int_{\{|Du_n|>m\}} |\mathbb{A}_n|\, \mathrm{d}x \right)^{\frac{1}{2}} + k^2 \int_{\{|Du_n|>m\}} |\mathbb{A}_n|\, \mathrm{d}x$$
$$+\, \gamma\, k\, \|u_n\|_{H^1(\Omega)^M} \big|\{|Du_n| > m\}\big|^{\frac{1}{2}} + \gamma\, k^2 \big|\{|Du_n| > m\}\big|. \tag{3.13}$$

Also note that

$$\sup_{n\in\mathbb{N}} \big|\{|Du_n| > m\}\big| \le \frac{1}{m} \sup_{n\in\mathbb{N}} \int_\Omega |Du_n|\, \mathrm{d}x \le \frac{C}{m},$$

hence $\sup_{n\in\mathbb{N}} \big|\{|Du_n| > m\}\big| \to 0$ as $m \to \infty$. Then, since by (2.2) $\mathbb{A}_n$ is equi-integrable in $L^1(\Omega)^{(M\times N)^2}$, we can pass to the limsup successively in n and in m in inequality (3.13) to get

$$\lim_{n\to\infty} \int_\Omega (\mathbb{A}_n - \mathbb{B}_n)\, D\bar{u}_n^k : \big(Du_n - D\bar{u}_n^k\big)\, \mathrm{d}x = 0. \tag{3.14}$$

Finally, (3.7) combined with (3.8) and (3.14) yields

$$\limsup_{n\to\infty} \left(\int_\Omega \mathbb{A}_n \big(Du_n - D\bar{u}_n^k\big) : \big(Du_n - D\bar{u}_n^k\big)\, \mathrm{d}x \right) \le c\, k^{1-\frac{p}{2}}, \quad \forall\, k \ge 1, \tag{3.15}$$

which implies the double limit (3.6) since $p > 2$.

Second step: Determination of the limit of the flux $\mathbb{A}_n Du_n$.

On the one hand, by (2.3), the Cauchy–Schwarz inequality and the estimates (2.2), (2.6) we have

$$\int_\Omega \big|\mathbb{A}_n \big(Du_n - D\bar{u}_n^k\big)\big|\, \mathrm{d}x$$
$$\le c \int_\Omega |\mathbb{A}_n|^{\frac{1}{2}} \big(\mathbb{A}_n \big(Du_n - D\bar{u}_n^k\big) : \big(Du_n - D\bar{u}_n^k\big)\big)^{\frac{1}{2}}\, \mathrm{d}x$$
$$\le c \left(\int_\Omega |\mathbb{A}_n|\, \mathrm{d}x \right)^{\frac{1}{2}} \left(\int_\Omega \mathbb{A}_n \big(Du_n - D\bar{u}_n^k\big) : \big(Du_n - D\bar{u}_n^k\big)\, \mathrm{d}x \right)^{\frac{1}{2}}$$
$$\le c \left(\int_\Omega \mathbb{A}_n \big(Du_n - D\bar{u}_n^k\big) : \big(Du_n - D\bar{u}_n^k\big)\, \mathrm{d}x \right)^{\frac{1}{2}}. \tag{3.16}$$

This combined with (3.6) yields

$$\lim_{k\to\infty} \left[\limsup_{n\to\infty} \int_\Omega \big|\mathbb{A}_n \big(Du_n - D\bar{u}_n^k\big)\big|\, \mathrm{d}x \right] = 0. \tag{3.17}$$

On the other hand, using the functional ellipticity (2.1) and the Hölder inequality combined with estimate (3.5), we have

$$\int_\Omega |Du_n - D\bar{u}_n|^2\, \mathrm{d}x$$
$$\le 2 \int_\Omega \big|Du_n - D\bar{u}_n^k\big|^2\, \mathrm{d}x + 2 \int_{\{\bar{u}_n^k \ne \bar{u}_n\}} \big|D\bar{u}_n^k - D\bar{u}_n\big|^2\, \mathrm{d}x$$
$$\le c \int_\Omega \mathbb{A}_n \big(Du_n - D\bar{u}_n^k\big) : \big(Du_n - D\bar{u}_n^k\big)\, \mathrm{d}x + c\, k^{2-p}, \tag{3.18}$$

which by (3.6) implies that

$$\lim_{n\to\infty}\int_\Omega |Du_n - D\bar{u}_n|^2\,\mathrm{d}x = 0. \tag{3.19}$$

Now, consider the decomposition of the flux

$$\mathbb{A}_n Du_n = \mathbb{B}_n D\bar{u}_n + \mathbb{A}_n\left(Du_n - D\bar{u}_n^k\right) + \left(\mathbb{A}_n - \mathbb{B}_n\right) D\bar{u}_n^k + \mathbb{B}_n\left(D\bar{u}_n^k - D\bar{u}_n\right), \tag{3.20}$$

which by (2.6), (3.4), (3.5) gives

$$\begin{aligned}&\int_\Omega |\mathbb{A}_n Du_n - \mathbb{B}_n D\bar{u}_n|\,\mathrm{d}x \\ &\le \int_\Omega \left|\mathbb{A}_n\left(Du_n - D\bar{u}_n^k\right)\right|\mathrm{d}x + k\int_\Omega |\mathbb{A}_n - \mathbb{B}_n|\,\mathrm{d}x + c\,k^{1-p},\end{aligned} \qquad \forall\, k \ge 1. \tag{3.21}$$

This combined with the double limit (3.17) and (2.2) yields

$$\lim_{n\to\infty}\int_\Omega |\mathbb{A}_n Du_n - \mathbb{B}_n D\bar{u}_n|\,\mathrm{d}x = 0. \tag{3.22}$$

Let us conclude. By limit (3.19) u_n and $\bar{u}_n$ actually converge weakly in $H_0^1(\Omega)^M$ to the same limit u. Moreover, thanks to the H-convergence of $\mathbb{B}_n$ to $\mathbb{B}$ and the associated convergence of the flux (3.2), the sequence $\mathbb{B}_n D\bar{u}_n$ converges to $\mathbb{B}Du$ weakly in $L^2(\Omega)^{M\times N}$. This combined with (3.22) implies that the sequence $\mathbb{A}_n Du_n$ also converges to $\mathbb{B}Du$ weakly in $L^1(\Omega)^{M\times N}$. Hence, we deduce the convergences (2.11) and the limit equation (2.12). Note that the convergences (2.11) only depend on the sequence n for which $\mathbb{B}_n$ H-converges. Moreover, the limit equation (2.12) only depends on the H-limit of $\mathbb{B}_n$. Therefore, Theorem 2.3 is proved. □

In the Proof of Theorem 2.3 we have used the following extension to systems of the celebrated Meyers L^p-estimate [26]. The proof follows the scheme of [5] with an adaptation due to the functional ellipticity (2.1). We will give a sketch of the proof to illuminate this point.

Proposition 3.1 (Meyers L^p-estimate)**.** *Let $\alpha, \beta > 0$ and let Ω be a regular bounded open set of $\mathbb{R}^N$. There exist a number $p > 2$ and a constant $C > 0$ which only depend on α, β and Ω, such that for any $\mathbb{B}$ tensor-valued function satisfying the conditions (2.1), (2.6), and for any $f \in W^{-1,p}(\Omega)^M$, the solution $u \in H_0^1(\Omega)^M$ of*

$$-\operatorname{Div}(\mathbb{B}Du) = f \quad \text{in } \Omega. \tag{3.23}$$

satisfies the estimate

$$\|u\|_{W^{1,p}(\Omega)^M} \le C\,\|f\|_{W^{-1,p}(\Omega)^M}. \tag{3.24}$$

Proof. Consider the decomposition of [5] in the proof of Theorem 4.3:

$$\frac{1}{\gamma + c}\,\mathbb{B} = \mathbb{B}_1 + \mathbb{B}_2 \quad \text{with} \quad \mathbb{B}_1 := \frac{1}{\gamma + c}\left(\mathbb{B}^s + c\,I\right) \text{ and } \mathbb{B}_2 := \frac{1}{\gamma + c}\left(\mathbb{B}^a - c\,I\right), \tag{3.25}$$

where $\mathbb{B}^s$ is the symmetric part of $\mathbb{B}$, $\mathbb{B}^a$ is the antisymmetric part of $\mathbb{B}$, $\gamma > 0$ is given by (2.6) and $c > 0$. We have

$$\begin{cases} \displaystyle\int_\Omega \mathbb{B}_1 Du : Du\,\mathrm{d}x \ge \mu\int_\Omega |Du|^2\,\mathrm{d}x, \quad \forall\, v \in H_0^1(\Omega)^M \\ |\mathbb{B}_1| \le 1,\ |\mathbb{B}_2| \le \nu \quad \text{a.e. in } \Omega, \end{cases} \tag{3.26}$$

where by choosing $c > \frac{\gamma^2 - \alpha^2}{2\alpha}$,

$$0 < \nu := \frac{\sqrt{\gamma^2 + c^2}}{\gamma + c} < \mu := \frac{\alpha + c}{\gamma + c} < 1. \tag{3.27}$$

Following [5] the solution u of (3.23) satisfies

$$u + R\,u = \frac{1}{\gamma + c}\,(-\Delta)^{-1} f \quad \text{where} \quad R := (-\Delta)^{-1}\,(\Delta + T_1 + T_2)\,,\ T_i := -\operatorname{Div}(\mathbb{B}_i D\cdot)\,. \tag{3.28}$$

First, let us estimate the bound of the operator R from $H^1_0(\Omega)^M$ into $H^1_0(\Omega)^M$. Since Δ is an isometry from $H^1_0(\Omega)^M$ onto $H^{-1}(\Omega)^M$, we have by (3.26)

$$\begin{aligned} \|R\|_{H^1_0(\Omega)^M} &\le \big\|(-\Delta)^{-1}\|_{H^1_0(\Omega)^M, H^{-1}(\Omega)^M} \Big(\|\Delta + T_1\|_{H^1_0(\Omega)^M, H^{-1}(\Omega)^M} + \|T_2\|_{H^1_0(\Omega)^M, H^{-1}(\Omega)^M}\Big) \\ &= \|\Delta + T_1\|_{H^1_0(\Omega)^M, H^{-1}(\Omega)^M} + \nu. \end{aligned} \tag{3.29}$$

Moreover, noting that $\Delta + T_1$ is a self-adjoint operator we have again by (3.26)

$$\|\Delta + T_1\|_{H^1_0(\Omega)^M, H^{-1}(\Omega)^M} = \sup_{v \in H^1_0(\Omega)^M \setminus \{0\}} \left| \frac{\int_\Omega (I - \mathbb{B}_1) Dv : Dv \,\mathrm{d}x}{\int_\Omega |Dv|^2 \,\mathrm{d}x} \right| \le 1 - \mu. \tag{3.30}$$

Therefore, we obtain that

$$\|R\|_{H^1_0(\Omega)^M} \le 1 - \mu + \nu < 1. \tag{3.31}$$

Next, let us estimate the bound of the operator R from $W^{1,p}_0(\Omega)^M$ into itself, for $p > 2$. At this level, the proof is different from the one of [5]. Let S be the operator defined by

$$S : L^p(\Omega)^{M\times N} \to W^{1,p}_0(\Omega)^M, \quad Sh := \Delta^{-1}\big(\operatorname{Div}(h)\big). \tag{3.32}$$

Then, denoting by D the derivative operator we have that $D \circ R \circ S$ is a linear operator which maps continuously $L^p(\Omega)^{M\times N}$ into itself. By the Riesz–Thorin theorem (see, *e.g.*, [22] Sect. 10.11) we have for $p, q \in (2, \infty)$ with $p < q$,

$$\|D \circ R \circ S\|_{L^p(\Omega)^{M\times N}} \le \|D \circ R \circ S\|^{\theta}_{L^2(\Omega)^{M\times N}} \|D \circ R \circ S\|^{1-\theta}_{L^q(\Omega)^{M\times N}} \quad \text{with} \quad \frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{q}, \tag{3.33}$$

which fixing $q > 2$ and using (3.31) implies that

$$\begin{aligned} \limsup_{p\to 2,\, p>2} \|D \circ R \circ S\|_{L^p(\Omega)^{M\times N}} &\le \|D \circ R \circ S\|_{L^2(\Omega)^{M\times N}} \\ &= \|R \circ S\|_{L^2(\Omega)^{M\times N}, H^1_0(\Omega)^M} \le \|R\|_{H^1_0(\Omega)^M} < 1. \end{aligned} \tag{3.34}$$

We also have for any $v \in W^{1,p}_0(\Omega)^M$,

$$\begin{aligned} \|R\,v\|_{W^{1,p}_0(\Omega)^M} &= \|D(R\,v)\|_{L^p(\Omega)^{M\times N}} \\ &= \|\,(D \circ R \circ S)\,Dv\|_{L^p(\Omega)^{M\times N}} \le \|D \circ R \circ S\|_{L^p(\Omega)^{M\times N}} \|v\|_{W^{1,p}_0(\Omega)^M}. \end{aligned} \tag{3.35}$$

Therefore, estimates (3.34) and (3.35) give $\|R\|_{W^{1,p}_0(\Omega)^M} < 1$ for p close to 2, which combined with equation (3.28) concludes the proof. □

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