# Phase Retrieval Procedure for Microwave Linear Arrays


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***Abstract*—A methodology to solve the phase retrieval problem arising in microwave linear array is proposed. The goal is to recover the complex array excitations from phaseless measurements of the far field. An approach combining convex optimization (to solve the phase retrieval problem) and two measurement runs (to mitigate the ambiguity problem) has been developed and numerically assessed in various representative examples. These results show that under appropriate conditions of noise and sampling, it is possible to uniquely retrieve the complex excitations of linear arrays from phaseless measurements.**



***Index Terms*—phase retrieval, antenna measurements, phaseless measurements, convex optimization.**


## I. Introduction

In microwaves, measuring accurately the phase may be a costly and difficult task in particular at high frequencies. There is therefore an important interest in developing amplitude-only measurement methods and several ones have already been proposed for antenna metrology [1]–[3], array diagnosis and imaging [4]–[6]. Phaseless measurements are indeed attractive because they require less accurate positioning systems and receivers (such as detectors or scalar network analyzers) that are much cheaper than those needed for amplitude and phase measurements.

Mathematically, the phase retrieval problem is notoriously very challenging. On the one hand, it requires the development of efficient and reliable reconstruction algorithms. On the other hand, the phase retrieval problem is often ill-posed since its solution is, in general, not unique.

In this paper, we focus our investigation on the phase retrieval for microwave 1-D (linear) arrays. Our objective is to retrieve the complex array element excitations from its radiated far field magnitude measurements. An efficient approach combining convex programming (to solve the phase retrieval problem) and two measurement runs (to mitigate the non uniqueness of the solution) is proposed. The goal is to empirically show that under appropriate conditions (of sampling and noise level), it is possible to uniquely recover the excitations of 1-D array from phaseless far field data.

## II. Phase Retrieval Problem Formulation

The phase retrieval seeks to reconstruct $N$ complex excitations $\boldsymbol{x}$ given only the magnitude of $M$ linear measurements $\boldsymbol{y}$. It can be formulated as follows:

$$\text{find } \boldsymbol{x} \;\; \text{subject to } |\boldsymbol{A}\boldsymbol{x}| = \boldsymbol{y} \tag{1}$$

where $\boldsymbol{A} \in \mathbb{C}^{M \times N}$ is the sensing matrix, $\boldsymbol{x} \in \mathbb{C}^N$ and $\boldsymbol{y} \in \mathbb{R}^M$. The geometry of the investigated problem, i.e. the imaging of a linear array, is represented in Fig. 1. The problem to solve

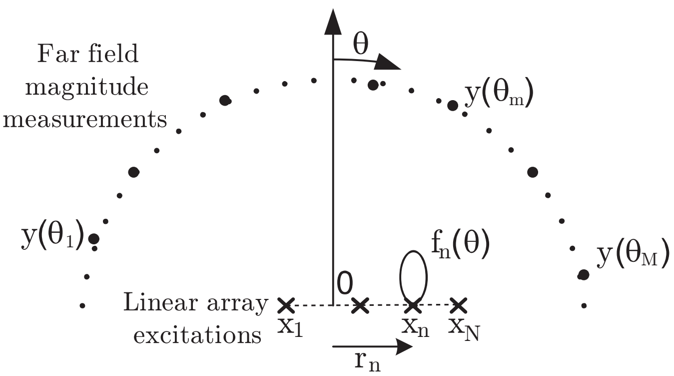


Fig. 1. Geometry of the linear array imaging problem with the notations.

can then be rewritten:

$$\text{find } \boldsymbol{x} \;\; \text{subject to } \left|\boldsymbol{a}_m{}^H \boldsymbol{x}\right|^2 = y_m^2, \;\text{ for } m = 1, ..., M \tag{2}$$

where $y_m = \boldsymbol{y}(\theta_m)$ and $(.)^H$ is the Hermitian transpose. The steering vector $\boldsymbol{a}(\theta_m)$ is denoted: $\boldsymbol{a}_m{}^H = [f_1(\theta_m)e^{j\frac{2\pi}{\lambda}r_1\sin\theta_m} \cdots f_N(\theta_m)e^{j\frac{2\pi}{\lambda}r_N\sin\theta_m}]$ where $\lambda$ is the free space wavelength, $r_n$ and $f_n(\theta_m)$ are the position and the far field pattern in the direction $\theta_m$ of the $n$-th antenna respectively.

The radiated field is thus related to the excitations by a discrete Fourier transform in the case of equi-spaced and isotropic sources. The phase retrieval problem is in general ill-posed since many different sets of excitations have the same Fourier transform magnitude. Moreover, it is difficult to solve because the set of real or complex numbers with a given magnitude is non-convex.

The ways (a) to solve the non-convex phase retrieval problem and (b) to mitigate the non-uniqueness (also called ambiguities) of the solution in order to uniquely recover the underlying excitations are addressed in Section III and IV respectively.

## III. Resolution via Convex Relaxation

The first convex relaxation of the phase retrieval problem has been introduced by Candès et al. [7]–[9]. He observed that the non-convex measurements $\boldsymbol{y}$ on vectors $\boldsymbol{x}$ become linear measurements on matrices $\boldsymbol{X} = \boldsymbol{x}\boldsymbol{x}^H$. The measurements can indeed be rewritten:

$$y_m^2 = \boldsymbol{x}^H \boldsymbol{A}_m \boldsymbol{x} = \mathrm{Tr}(\boldsymbol{A}_m \boldsymbol{X})$$

where $\boldsymbol{A}_m = \boldsymbol{a}_m \boldsymbol{a}_m{}^H$ are Hermitian matrices and $\boldsymbol{X} = \boldsymbol{x}\boldsymbol{x}^H$ is a rank-one Hermitian matrix.
The phase retrieval problem (2) becomes:

$$\begin{array}{ll} \text{find} & \boldsymbol{X} \\ \text{subject to} & \mathrm{Tr}(\boldsymbol{A}_m \boldsymbol{X}) = y_m^2, \;\; m = 1, ..., M \\ & \boldsymbol{X} \succeq 0 \\ & \mathrm{rank}(\boldsymbol{X}) = 1 \end{array} \tag{3}$$

that is equivalent to:

$$\begin{array}{ll} \text{minimize} & \mathrm{rank}(\boldsymbol{X}) \\ \text{subject to} & \mathrm{Tr}(\boldsymbol{A}_m \boldsymbol{X}) = y_m^2, \;\; m = 1, ..., M \\ & \boldsymbol{X} \succeq 0 \end{array} \tag{4}$$

since there exists by definition a rank-one solution.
The problem (4) is a combinatorially hard problem. However, for positive semidefinite matrices, i.e. in this case since $\boldsymbol{X} \succeq 0$, the rank functional can be approximated by a convex surrogate, the trace norm as proposed in [10]. The problem (4) becomes:

$$\begin{array}{ll} \underset{\boldsymbol{X}}{\text{minimize}} & \mathrm{Tr}(\boldsymbol{X}) \\ \text{subject to} & \mathrm{Tr}(\boldsymbol{A}_m \boldsymbol{X}) = y_m^2, \;\; m = 1, ..., M \\ & \boldsymbol{X} \succeq 0 \end{array} \tag{5}$$

which is a semidefinite program that is convex and therefore efficiently solvable. The original vectorial phase retrieval problem is thus convexified by "lifting" it up to a matrix recovery problem hence the name *PhaseLift* given in [7], [8].
In practice, the measurements are contaminated by noise:

$$y_m = |\boldsymbol{a}_m{}^H \boldsymbol{x} + n_m|, \text{ for } m = 1, \cdots, M$$

where $n_m$ is a noise term. The equalities in (5) no longer hold in presence of noise. The following formulation has then been proposed in [9]:

$$\begin{array}{ll} \underset{\boldsymbol{X}}{\text{minimize}} & \displaystyle\sum_{m=1}^{M} \left| \, \mathrm{Tr}(\boldsymbol{A}_m \boldsymbol{X}) - y_m^2 \, \right| \\ \text{subject to } \boldsymbol{X} \succeq 0. & \end{array} \tag{6}$$

In words, solving (6) amounts to find the positive semidefinite matrix $\boldsymbol{X}$ that best fits the observed data in an $\ell_1$ sense.
If the solution $\widehat{\boldsymbol{X}}$ of (6) happens to have rank one (this is not the case in general), then $\widehat{\boldsymbol{X}} = \hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^H$ and $\hat{\boldsymbol{x}}$ is the optimal solution of the original phase retrieval problem (2). Otherwise, one extracts the best rank one approximation of $\widehat{\boldsymbol{X}}$ by taking its top eigen-pair $(\lambda_1, \boldsymbol{u}_1)$, where $\lambda_1$ is the largest eigenvalue of $\widehat{\boldsymbol{X}}$ and $\boldsymbol{u}_1$ is the associated eigenvector. The vector $\hat{\boldsymbol{x}} = \sqrt{\lambda_1}\boldsymbol{u}_1$ is then an approximate solution of (2).
The convex relaxation problem (6) can be optimally and efficiently solved by freely available software such as CVX [11]. Of course, there is a price to pay in turning a problem that is originally hard to solve into an efficiently solvable one. The lifting procedure transforms a vector into a matrix problem which implies a much larger representation of the state space and consequently a higher computational cost.

## IV. Application to Array Imaging Problems

### A. Non-uniqueness of the Solution

The solution to the phase retrieval problem (1) is, in general, not unique and many approaches have already been proposed in the literature to mitigate the potential ambiguities. In order to be easily applicable to microwave measurements, we use a procedure that is similar to what is done in holography [12]. Let us assume that we have a reference antenna whose complex radiation pattern $y^{ref}(\theta)$ (the vector $\boldsymbol{y}^{ref}$ of dimension $M$ after discretization) is known. We measure the magnitude of the far field radiated by:
- the array under test $\boldsymbol{y}^{AUT} = |\boldsymbol{A}^{AUT}\boldsymbol{x}|$ and
- the interference between the array under test and the reference antenna $\boldsymbol{y}^{AUT+ref} = |\boldsymbol{A}^{AUT}\boldsymbol{x} + \alpha \boldsymbol{y}^{ref}|$
where $\alpha$ is the excitation (not necessarily known) of the reference antenna.
This simple procedure allows to mitigate the ambiguities and the excitations are then determined up a global phase. If in addition the phase of the excitation $\alpha$ is known (after a calibration for instance), then the solution $\boldsymbol{x}$ is unique.

### B. Recovery Performances

*1) Recovery Error:* The phase retrieval algorithm (6) computes an approximate solution $\hat{\boldsymbol{x}}$ from $\boldsymbol{y} = |\boldsymbol{A}\boldsymbol{x}|$. Its performance is assessed by calculating the relative "distance" between the exact solution $\boldsymbol{x}$ and the recovered one $\hat{\boldsymbol{x}}$. Special care must be taken since a solution may be unique up to a global phase. Thus, the error in excitation denoted $\epsilon(\boldsymbol{x}, \hat{\boldsymbol{x}})$ can be computed as:

$$\frac{\|\boldsymbol{x}\boldsymbol{x}^H - \hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^H\|_F}{\|\boldsymbol{x}\boldsymbol{x}^H\|_F} \tag{7}$$

where $\|.\|_F$ stands for the Frobenius norm. The error over the measured field amplitudes is:

$$\epsilon(|\boldsymbol{A}\boldsymbol{x}|, |\boldsymbol{A}\hat{\boldsymbol{x}}|) = \frac{\| \, |\boldsymbol{A}\boldsymbol{x}| - |\boldsymbol{A}\hat{\boldsymbol{x}}| \, \|_2}{\|\boldsymbol{A}\boldsymbol{x}\|_2}. \tag{8}$$

In order to appreciate the results with more physical insights, we can also compute the mean values of the excitation amplitude ratio and phase difference (denoted $\mu_m(\boldsymbol{x}, \hat{\boldsymbol{x}})$ and $\mu_p(\boldsymbol{x}, \hat{\boldsymbol{x}})$ respectively):

$$\begin{aligned} \mu_m(\boldsymbol{x}, \hat{\boldsymbol{x}}) &= \frac{1}{N} \sum_{i=1}^{N} \delta_{mi} \text{ with } \delta_{mi} = \frac{|\hat{\boldsymbol{x}}_i|}{|\boldsymbol{x}_i|} \\ \mu_p(\boldsymbol{x}, \hat{\boldsymbol{x}}) &= \frac{1}{N} \sum_{i=1}^{N} |\delta_{pi}| \text{ with } \delta_{pi} = (\angle \hat{\boldsymbol{x}}_i - \Delta\phi) - \angle \boldsymbol{x}_i \end{aligned} \tag{9}$$

where $\Delta\phi$ the global phase shift between $\boldsymbol{x}$ and $\hat{\boldsymbol{x}}$ is equal to $\Delta\phi = \frac{1}{N} \sum_{i=1}^{N} (\angle \hat{\boldsymbol{x}}_i - \angle \boldsymbol{x}_i)$.

We derive from (9) the standard deviation in amplitude and phase ($\sigma_m(\boldsymbol{x},\hat{\boldsymbol{x}})$ and $\sigma_p(\boldsymbol{x},\hat{\boldsymbol{x}})$ respectively):

$$\sigma_m(\boldsymbol{x},\hat{\boldsymbol{x}}) = \left[\frac{1}{N}\sum_{i=1}^{N}\left(\delta_{mi} - \mu_m(\boldsymbol{x},\hat{\boldsymbol{x}})\right)^2\right]^{1/2} \quad (10)$$

$$\sigma_p(\boldsymbol{x},\hat{\boldsymbol{x}}) = \left[\frac{1}{N}\sum_{i=1}^{N}\left(\delta_{pi} - \mu_p(\boldsymbol{x},\hat{\boldsymbol{x}})\right)^2\right]^{1/2}$$

in order to measure the degree of confidence in the retrieved excitation magnitudes and phases.

*2) Noise:* The reconstruction of the excitations in presence of noise is crucial for practical applications. In our experiments, the magnitude measurement $y_m$ is polluted by a Gaussian white noise $n_m$ as follows: $y_m = |\boldsymbol{a}_m{}^H \boldsymbol{x} + n_m|$. The level of this noise is quantified by the Signal-to-Noise Ratio (SNR): $\text{SNR}_{dB} = 10\log_{10}(P_{signal}/P_{noise})$ where $P_{signal} = \max_{m=1,\cdots,M}(|\boldsymbol{a}_m{}^H \boldsymbol{x}|^2)$ is the maximum measured power.
In order to estimate the SNR of a far field measurement in an anechoic chamber, both the full equipment system (noise floor of the receiver, transmitted power and losses due to the cables and rotary joints) and the reflectivity of the chamber itself must be taken into account. A SNR value of 60 dB, that corresponds to a reasonably good anechoic chamber, will be considered for the numerical applications of Section V.

## V. Numerical Results

The goal of this Section is to numerically assess the recovery performances of the phase retrieval algorithm (6). In the retrieval procedure, the element positions and radiation patterns of the array under test, i.e. of the sensing matrix $\boldsymbol{A}$, and the reference antenna $y^{ref}$ are assumed known.
For each configuration, the simulation is repeated 100 times with different random excitations (whose magnitude and phase are chosen uniformly between $[0,1]$ and $[0,2\pi]$ respectively), different random measurement vectors (whose angle are chosen uniformly such that $-1 \leq \sin\theta \leq 1$) and a Gaussian white noise in order to get meaningful results. All presented results are average values over these 100 simulations.

### *A. Linear Array of Isotropic Sources*

*1) Settings:* We consider a linear array composed of $N$ isotropic half wavelength spaced elements that are excited by a random amplitude and phase $x_n$. A reference antenna (also isotropic) with an arbitrary excitation (fixed but chosen randomly in the simulations) is placed at an arbitrary distance ($2\,\lambda$ in the simulations) from the extremity of the array under test. It has been checked numerically that both the position and excitation value of this source with respect to the array under test do not impact the recovery performances.

*2) Influence of the Sampling:* The influence of the ratio $M/N$ (number of measurement points over number of excitations to retrieve) on the recovery performances is investigated. The error in excitations $\epsilon(\boldsymbol{x},\hat{\boldsymbol{x}})$ is plotted as a function of the sampling for linear arrays of various elements $N$ in Fig. 2. In absence of noise, there is a clear transition between 'bad' and 'good' recovery at $M = 2N$. In presence of a realistic noise (SNR=60 dB), a less sharp transition also exists around $M = 2N$. Generally speaking, for a given sampling $M/N$, the recovery performances are better for a small number of array elements $N$.

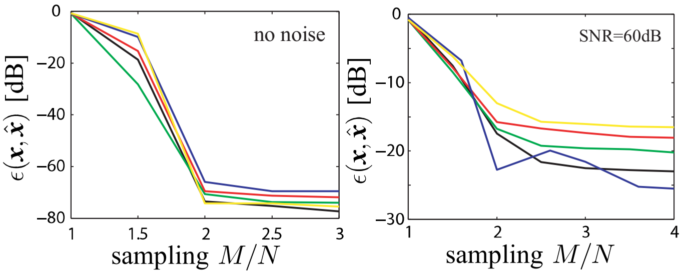


Fig. 2. Error in the retrieved excitation amplitudes as a function of the sampling for linear array of various elements ($N$=5, 10, 20, 30, 40).

To evaluate the reconstruction quality as a function of the sampling, let us plot in Fig. 3 the reconstruction errors in amplitude and phase for a linear array of $N$=10 elements in presence of noise (SNR=60 dB). We retrieve, as seen above, the significant improvement in recovery performances when $M = 2N$. Then, the oversampling helps to further improve up to a certain extent the recovery performances as shown in Fig. 3 for $M > 2N$.

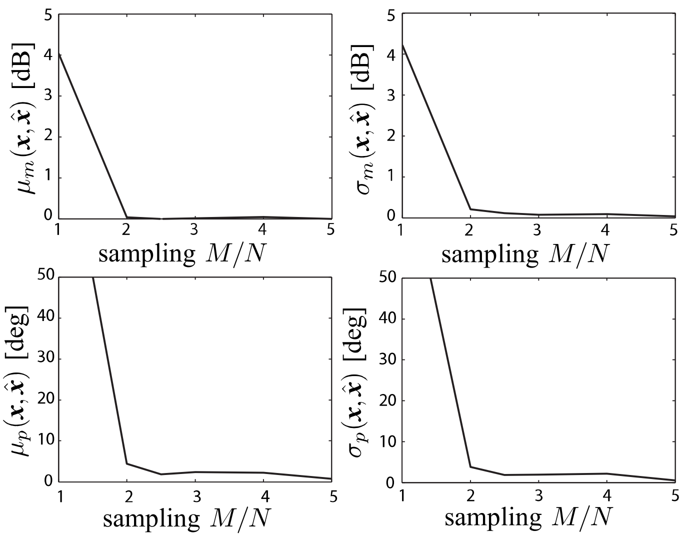


Fig. 3. Recovery performances for a linear array of $N$=10 isotropic sources: influence of the sampling ($M/N$) for measurements with a SNR of 60 dB.

*3) Robustness to Noise:* Let us choose a sampling $M = 2N$ in order to see the influence of the noise on the excitation recovery.
The influence of the SNR on the retrieved excitations and field amplitudes ($\epsilon(\boldsymbol{x},\hat{\boldsymbol{x}})$ and $\epsilon(|\boldsymbol{A}\boldsymbol{x}|,|\boldsymbol{A}\hat{\boldsymbol{x}}|)$ respectively) is plotted in Fig. 4. There is clearly a linear behavior between these errors and the SNR with a log-log scale. This graceful degradation shows that the convex approach provides a stable recovery in

presence of noise.

The reconstruction errors in excitation amplitude and phase

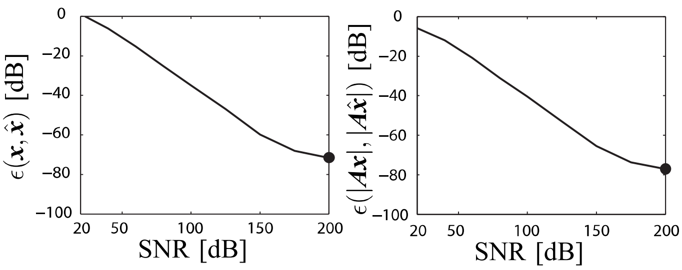


Fig. 4. Error in the retrieved excitations and field amplitudes for a linear array of $N$ =10 elements as a function of the SNR. The markers are the results without noise.

are plotted in Fig. 5 as a function of the SNR. They show that above a given SNR (of 60 dB for $M/N = 2$), the reconstruction is very good in amplitude and fairly good in phase.

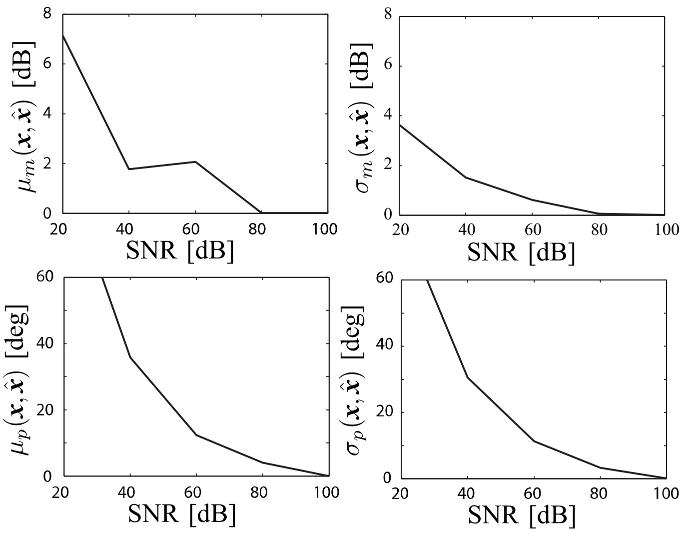


Fig. 5. Recovery performances for a linear array of $N$=10 isotropic sources with a sampling $M = 2N$: influence of the SNR.

## VI. Conclusion

A procedure has been proposed to efficiently retrieve the complex excitations of microwave linear arrays from amplitude-only measurements. Promising results stable with respect to noise have been obtained. They must be confirmed by experimental validations. More details and numerical application examples about the proposed approach will be given at the conference and can be found in [13].

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