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# Array Pattern Synthesis with Excitation Control via Norm Minimization

Benjamin Fuchs, *Senior Member, IEEE* and Sébastien Rondineau

**Abstract**—Simple and efficient procedures are presented to synthesize antenna array patterns (focused or shaped beams) while controlling the excitations. This synthesis problem is of great practical interest as it makes active arrays more attractive by reducing their complexity, cost and weight. With a proper regularization of the array pattern synthesis solution, the excitations can be optimized in order to design uniform amplitude sparse arrays, arrays whose excitations have a low dynamic range ratio or a smooth amplitude variation. More specifically, the synthesis procedures are composed of the minimization of a norm (mixed  $\ell_1/\ell_\infty$  norm, total variation norm or a combination of the  $\ell_1$ - and  $\ell_\infty$ -norms) associated with radiation pattern constraints. Various representative array synthesis problems are considered. Examples including the coupling between antennas and a comparison with a deterministic approach confirm both the usefulness and effectiveness of the proposed procedures.

**Index Terms**—Antenna arrays, antenna synthesis, phase-only control, convex optimization.

## I. INTRODUCTION

THE synthesis of antenna arrays with controlled excitations is an issue of increasing relevance, especially for embedded applications such as satellite and unmanned aerial vehicles [1], [2]. It enables to reduce the complexity, cost and weight of active arrays and consequently make them an attractive alternative to the commonly used reflector antennas to generate the multiple beams satellite coverage.

In addition to radiation pattern requirements, the antenna array excitations are optimized in the synthesis procedure in order to, for instance, simplify the beamforming network. Thus the synthesis of uniform amplitude sparse arrays [3]–[6] and arrays whose excitations have a reduced Dynamic Range Ratio (DRR) [7]–[10] have recently drawn a lot of attention.

There are several benefits in designing antenna arrays having equi-amplitude (isophoric) excitations as well explained in [3] and [11]. Generally speaking, reducing the number of control points (amplifiers and phase shifters) enables to make active arrays more attractive in terms of cost, reliability and power efficiency.

The price to pay in adding constraints on the array excitations is the degradation of the radiating performances unless the antenna locations are also optimized. The excitation taper required to fulfill the pattern constraints is then replaced by a spatial taper of the elements leading to non uniformly spaced arrays.

Deterministic approaches have been proposed to efficiently

synthesize uniform amplitude sparse arrays radiating pencil beams in [3]–[5] and more recently shaped beams [6]. These efficient methods are based on the density taper strategy introduced by [12] and they outperform in canonical configurations approaches using so-called global optimizers that are versatile but computationally heavy [13], [14].

Besides, arrays whose excitations exhibit a low dynamic range ratio or a smooth variation are also of great practical interest because they enable a better control of the mutual coupling between antennas. A projection based approach [7] and fast iterative methods [9], [10] have been proposed in to synthesize complex radiation patterns with various types of arrays while simultaneously reducing the DRR of the excitations. The cost function includes then both the pattern to fit and the DRR.

In this paper, several procedures are presented to synthesize array pattern while controlling the excitations. With a proper regularization of the pattern synthesis solution (mostly via norm minimization), the array excitations can be optimized in order to design uniform amplitude sparse arrays, arrays with low DRR or smooth amplitude variation while taking the coupling between antennas into account. All proposed optimization schemes are convex and therefore easily and efficiently solvable on readily available solvers such as CVX [15]. However, it does not mean that all solutions are optimal because approximations are necessary, notably in case of shaped beam synthesis, in order to obtain convex formulations.

The paper is organized as follows. The array notations and the definition of norms are introduced in Section II. The way to formulate the synthesis of arrays with various excitation control is described in Section III. A set of representative numerical examples is provided in Section IV to both validate and illustrate the proposed approaches. Conclusions and perspectives are drawn in Section V.

## II. ARRAY NOTATIONS AND NORMS

### A. Array Notations

Let us consider an array composed of  $N$  antennas placed at locations  $\vec{r}_n$ . For the sake of clarity, the problem is described for a one-dimensional pattern synthesis, i.e. over the polar angle  $\theta$  in a fixed azimuthal plane  $\varphi = \varphi_0$  that is omitted in the notations. The extension to a two-dimensional pattern synthesis, i.e. a synthesis over both angular directions  $\theta$  and  $\varphi$ , is straightforward. Moreover, we consider the synthesis of one component of the electromagnetic field to simplify the notations. Each antenna  $n$  radiates a complex pattern  $g_n(\theta)$  in the direction  $\theta$ . The electromagnetic far field pattern  $f(\theta)$

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Benjamin Fuchs (benjamin.fuchs@univ-rennes1.fr) is with the University of Rennes I/ IETR, France and Sébastien Rondineau (sebastien@unb.br) is with the University of Brasilia, Brazil.

radiated by the antenna array is:

$$f(\theta) = \mathbf{a}(\theta)^H \mathbf{x}, \quad (1)$$

$$\text{with } \mathbf{a}(\theta) = [g_1(\theta)e^{j\frac{2\pi}{\lambda}r_1\hat{r}(\theta)} \dots g_N(\theta)e^{j\frac{2\pi}{\lambda}r_N\hat{r}(\theta)}]^H$$

where  $\mathbf{x}$  is the complex excitation vector,  $\hat{r}(\theta)$  is the unit vector in the direction  $\theta$  and  $\cdot^H$  denotes the Hermitian transposition. The power pattern radiated by the array can be expressed:

$$|f(\theta)|^2 = \mathbf{x}^H \mathbf{a}(\theta) \mathbf{a}(\theta)^H \mathbf{x} \quad (2)$$

$$= \text{Tr}(\mathbf{A}(\theta)\mathbf{X})$$

$$\text{with } \mathbf{A}(\theta) = \mathbf{a}(\theta)\mathbf{a}(\theta)^H \text{ and } \mathbf{X} = \mathbf{x}\mathbf{x}^H$$

where  $\text{Tr}(\mathbf{C})$  is the trace of the matrix  $\mathbf{C}$ . The matrix  $\mathbf{A}(\theta)$  is Hermitian and  $\mathbf{X} = \mathbf{x}\mathbf{x}^H$  is an Hermitian and positive semidefinite matrix (denoted  $\mathbf{X} \succeq 0$ ) of rank one. Thus the constraint  $|f(\theta)|^2 = \beta$  is equivalent to:

$$\text{Tr}(\mathbf{A}(\theta)\mathbf{X}) = \beta \text{ with } \mathbf{X} \succeq 0 \text{ and } \text{rank}(\mathbf{X}) = 1.$$

To make this constraint convex with respect to  $\mathbf{X}$  and therefore easily solvable, the rank constraint is dropped and replaced by the minimization of the trace of  $\mathbf{X}$ . Indeed, this convex relaxation amounts to minimize the sum of the eigenvalues of  $\mathbf{X}$  (all real and positive values) and therefore its rank.

The sought-for excitation vector  $\mathbf{x}$  is then extracted by taking the best rank one approximation of  $\mathbf{X}$  through eigenvalue decomposition:  $\mathbf{x} = \sqrt{\sigma_1}\mathbf{u}_1$ , where  $(\sigma_1, \mathbf{u}_1)$  is the top eigenpair of  $\mathbf{X}$ . This technique is known as semidefinite relaxation and more details can be found in [16], [17].

The use of the proxy matrix  $\mathbf{X}$  instead of the vector  $\mathbf{x}$  leads to a significant increase (from  $N$  to  $N^2$ ) of the number of variables. This is the price to pay to turn an initially NP-hard problem into a polynomial time solvable one.

### B. Array Pattern Synthesis

A focused beam pattern can be synthesized by applying the following constraints on the radiated field  $f(\theta)$ : a main beam radiated in the direction  $\theta_0$  with sidelobes below a given upper bound  $\rho(\theta)$  over an angular region  $\Theta_{sl}$ . These constraints denoted  $\mathcal{FB}$  reads:

$$\begin{cases} |f(\theta_0)| \geq 1 \\ |f(\theta)| \leq \rho(\theta), \forall \theta \in \Theta_{sl} \end{cases} \quad (3)$$

It yields, after discretization of  $\Theta_{sl}$ :

$$\mathcal{FB}(\mathbf{x}) \begin{cases} \Re(\mathbf{a}_0^H \mathbf{x}) \geq 1 \\ |\mathbf{a}_m^H \mathbf{x}| \leq \rho_m, \forall m \end{cases} \quad (4)$$

$$\text{with } \mathbf{a}_i = \mathbf{a}(\theta_i) \text{ and } \rho_m = \rho(\theta_m)$$

where the array excitations  $\mathbf{x}$  are the unknowns. Note that the constraint on the real part  $\Re(\mathbf{a}_0^H \mathbf{x}) \geq 1$  is convex and equivalent to the non convex one  $|\mathbf{a}_0^H \mathbf{x}| \geq 1$  in this case, since the excitations  $\mathbf{x}$  are determined up to a global phase shift.

A focused beam pattern  $\mathcal{FB}$  can also be achieved using (2) by solving:

$$\mathcal{FB}(\mathbf{X}) : \min_{\mathbf{X}} \text{Tr}(\mathbf{X}) \text{ s.t. } \begin{cases} \text{Tr}(\mathbf{A}_0\mathbf{X}) \geq 1 \\ \text{Tr}(\mathbf{A}_m\mathbf{X}) \leq \rho_m^2, \forall m \\ \mathbf{X} \succeq 0 \end{cases} \quad (5)$$

where ‘s.t.’ stands for ‘subject to’ and  $\mathbf{A}_i = \mathbf{A}(\theta_i)$ .

Let us point out that the constraints (5) are convex with respect to the unknown  $\mathbf{X}$  as well as the square magnitude of the excitations since:

$$\text{diag}(\mathbf{X}) = [|\mathbf{x}_1|^2 \dots |\mathbf{x}_N|^2]^T. \quad (6)$$

This interesting property will be used in Section III.

Let us define a (shaped) beam by either a complex pattern  $y^d(\theta)$  or a power pattern  $|y^d(\theta)|^2$  over a given angular range  $\Theta_{sb}$ . The synthesis of a desired (shaped) beam can be formulated by the constraints  $\mathcal{SB}$ :

$$|f(\theta) - y^d(\theta)| \leq \epsilon, \forall \theta \in \Theta_{sb} \quad (7)$$

$$\text{or } \left| |f(\theta)|^2 - |y^d(\theta)|^2 \right| \leq \epsilon, \forall \theta \in \Theta_{sb} \quad (8)$$

respectively. The parameter  $\epsilon$  is the bound setting the maximum ripple around the desired beam. It yields after discretization of  $\Theta_{sb}$  in the directions  $\theta_s$ :

$$\mathcal{SB}(\mathbf{x}) : |\mathbf{a}_s^H \mathbf{x} - y_s^d| \leq \epsilon, \forall s \quad (9)$$

$$\text{or } \mathcal{SB}(\mathbf{X}) : \min_{\mathbf{X}} \text{Tr}(\mathbf{X}) \text{ s.t. } \begin{cases} |\text{Tr}(\mathbf{A}_s\mathbf{X}) - |y_s^d|^2| \leq \epsilon, \forall s \\ \mathbf{X} \succeq 0 \end{cases} \quad (10)$$

### C. Norms

Let us recall a few definitions of norms that will be applied on the array excitations  $\mathbf{x}$  so as to regularize the solution of pattern synthesis problems and thereby synthesize arrays with controlled excitations.

The  $\ell_1$ -norm and the  $\ell_\infty$ -norm of a vector  $\mathbf{x} \in \mathbb{C}^N$  are:

$$\|\mathbf{x}\|_1 = \sum_{n=1}^N |x_n| \text{ and } \|\mathbf{x}\|_\infty = \max_{n \in \{1, \dots, N\}} |x_n|. \quad (11)$$

The minimization of the  $\ell_1$ -norm of a solution  $\mathbf{x}$  is known to promote sparse solutions [18]. It has been successfully used in a wide range of applications including sparse array synthesis [19], [20]. On the other hand, minimizing the  $\ell_\infty$ -norm of a vector limits the range of its components which in turn tend to be stuck in the limit, i.e.  $|x_n| = \|\mathbf{x}\|_\infty$ , as explained in [21]–[23].

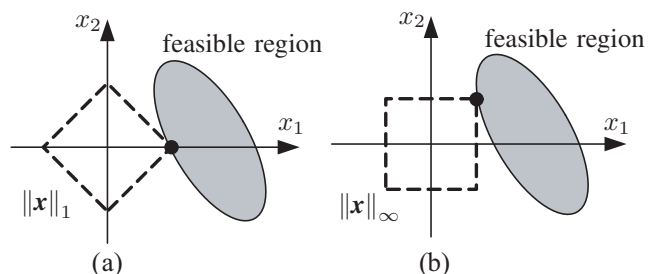


Fig. 1. Illustration of the  $\ell_1$ - and  $\ell_\infty$ -norm regularization. The solution is marked by a black dot. (a) The minimization of  $\|\mathbf{x}\|_1$  leads to a sparse solution:  $\|\mathbf{x}\|^* = (x_1^*, 0)$ . (b) The minimization of  $\|\mathbf{x}\|_\infty$  leads to a solution with equal magnitude components:  $x_1^* = x_2^*$ .

A 2D illustration of the  $\ell_1$ - and  $\ell_\infty$ - norms regularizing a solution of a feasible set is given in Fig. 1. Although simplified, this figure clearly illustrates the two extreme effects induced by these norms.

Let us now divide the vector  $\mathbf{x}$  into  $G$  non overlapping groups denoted  $\mathbf{x}^g$  of size  $N_g$ , such as  $\sum_{g=1}^G N_g = N$  and  $\mathbf{x} = \bigcup_{g=1, \dots, G} \mathbf{x}^g$ . The mixed  $\ell_1/\ell_\infty$  norm reads:

$$\|\mathbf{x}\|_{1, \infty} = \sum_{g=1}^G \|\mathbf{x}^g\|_\infty = \sum_{g=1}^G \max(|x_1^g|, \dots, |x_{N_g}^g|). \quad (12)$$

This norm is an instance of group (or structured) sparsity-inducing norm, as thoroughly described in [24]. Minimizing the  $\ell_1/\ell_\infty$ -norm has the same effect as minimizing the  $\ell_\infty$ -norm for each group (sub-vector)  $\mathbf{x}^g$ . It means that the magnitude of all components of the  $g$ -th group will tend to be equal to  $\|\mathbf{x}^g\|_\infty$ .

The Total Variation norm (TV-norm) is a smoothing function, introduced in [25], that is defined for real vectors  $\mathbf{x} \in \mathbb{R}^N$ . It can be expressed:

$$\|\mathbf{x}\|_{\text{TV}} = \sum_{n=2}^N |x_n - x_{n-1}| = \|\nabla \mathbf{x}\|_1 \quad (13)$$

$$\text{where } \nabla = \begin{bmatrix} -1 & 1 & & & 0 \\ & -1 & 1 & & \\ & & & \ddots & \\ 0 & & & & -1 & 1 \end{bmatrix}.$$

The matrix  $\nabla$  of size  $N - 1 \times N$  is a discrete gradient matrix. Minimizing the TV-norm of a real vector  $\mathbf{x}$  reduces the variation of its components hence the smoothing effect.

### III. ARRAY SYNTHESIS WITH EXCITATION CONTROL

The synthesis of antenna arrays with excitation control can be achieved by minimizing a norm or a functional of the solution in order to promote a type of excitation while fulfilling radiation pattern constraints. The optimization problems are of the form:

$$\min_{\mathbf{x}} \|\mathbf{x}\|_p \quad \text{s.t. } \mathcal{FB}(\mathbf{x}) \text{ or } \mathcal{SB}(\mathbf{x}) \quad (14)$$

$$\text{or } \min_{\mathbf{X}} f(\mathbf{X}) + \gamma \text{Tr}(\mathbf{X}) \quad \text{s.t. } \mathcal{FB}(\mathbf{X}) \text{ or } \mathcal{SB}(\mathbf{X}) \quad (15)$$

where the  $\ell_p$ -norm or convex functional  $f$  is chosen in order to synthesize arrays with the desired type of excitations.

The positive scalar  $\gamma$  in (15) balances the trade-off between the functional minimization and the low rank solution. This parameter  $\gamma$  is chosen such as  $\sigma_1 \gg \sigma_2$ , where  $\sigma_i$  is the  $i$ -th eigenvalue of  $\mathbf{X}$ , in order to ensure a low rank solution  $\mathbf{X}$ .

#### A. Sparse Arrays with Uniform Amplitude and Phase Excitations

The  $\ell_1$ -norm and  $\ell_\infty$ -norm minimizers are two extremes to regularize the solution of an optimization problem, as explained in Section II. However, when appropriately combined, these two norms can promote the underlying binary sparsity of a solution and therefore be used to synthesize sparse arrays

whose excitations have the same amplitude and phase. Thus, isophoric arrays can be synthesized by solving:

$$\min_{0 \leq x \leq 1} \|\mathbf{x}\|_1 + \gamma \|\mathbf{x} - \alpha\|_\infty \quad \text{s.t. } \mathcal{FB}(\mathbf{x}) \text{ or } \mathcal{SB}(\mathbf{x}) \quad (16)$$

Let us explain the role of the scalar  $\alpha$ . We assume that there is a binary prior on the excitations ( $x_n \in \{0, 1\}$ ), i.e. there exists an isophoric array that satisfies the pattern constraints  $\mathcal{FB}(\mathbf{x})$  or  $\mathcal{SB}(\mathbf{x})$ . All excitations  $x_n$  are then of equal distance to 0.5. Consequently, if  $\alpha = 0.5$ , minimizing  $\max |x_n - 0.5|$  will tend to stuck the components of  $x_n$  at the values 0 or 1. Thus, the sparsity is promoted by the  $\ell_1$ -norm penalty whereas the binary property is fostered by the  $\ell_\infty$ -norm term which encourages the non-zero coefficients to stick to 1.

The parameter  $\gamma$  in (16) controls the trade-off between the sparsity and the binary aspect of the solution. Its choice is a notoriously difficult problem arising in many optimization problems. One solution is to solve the problem (16), that is computationally cheap, multiple times for different values of  $\gamma$ . Otherwise, it has been empirically noticed that a ‘good’ parameter  $\gamma$  balances equally the  $\ell_1$ - and  $\ell_\infty$ -norms. It means that if we know a priori that the isophoric arrays is composed of  $N$  elements, then  $\gamma$  should be around  $2N$ .

To further enhance the sparsity of the solution, the reweighted  $\ell_1$  minimization algorithm introduced in [26] is used. At iteration  $k = 0$ , we start from a fully populated array and solve (16). Then, the weakest excitations are removed (e.g. the  $x_n$ 's such as  $|x_n| < \max_n 0.001 |x_n|$ ) and we solve the following reweighted minimization problem:

$$\begin{aligned} & \min_{\mathbf{x}^k} \|\mathbf{W}^k \mathbf{x}^k\|_1 + \gamma \|\mathbf{x}^k - \alpha\|_\infty \quad \text{s.t. } \mathcal{FB}(\mathbf{x}^k) \text{ or } \mathcal{SB}(\mathbf{x}^k) \\ & \text{with } \begin{cases} \mathbf{W}_{n,n}^k = (|x_n^{k-1}| + \delta)^{-1} \\ \mathbf{W}_{n,p}^k = 0 \quad \text{if } n \neq p \end{cases} \end{aligned} \quad (17)$$

The matrix  $\mathbf{W}^k$  is a diagonal weighting matrix that penalizes heavily small excitations (see [26]) in order to foster more aggressively sparse solutions. The parameter  $\delta > 0$  ensures the numerical stability of algorithm. At each iteration  $k \geq 1$ , we solve (17), remove the lowest excitations and so on until the number of zero excitation  $\mathbf{x}$  do not diminishes anymore.

#### B. Groups with Uniform Magnitude Excitations

The use of the mixed  $\ell_1/\ell_\infty$ -norm enables to synthesize arrays with  $G$  predefined groups of antennas having the same excitation magnitudes. Such synthesis problem can be formulated:

$$\min_{\mathbf{x}} \sum_{g=1}^G \|\mathbf{x}^g\|_\infty \quad \text{s.t. } \mathcal{FB}(\mathbf{x}) \text{ or } \mathcal{SB}(\mathbf{x}). \quad (18)$$

It is important to note that the element locations and the groups of elements are fixed a priori but both the phases of the elements and the excitation amplitudes within each group are left free and optimized when solving (18).

### C. Excitations with Minimized Dynamic Range Ratio

The Dynamic Range Ratio (DRR) of the excitation vector  $\mathbf{x}$  is the ratio between the maximum and minimum excitation magnitude:  $\text{DRR}(\mathbf{x}) = \max_n |x_n| / \min_n |x_n|$ .

When the excitations are real values, the array synthesis with a minimized DRR amounts to solve the convex problem:

$$\begin{aligned} \min_{\mathbf{x} \in \mathbb{R}^N} \quad & \max(\mathbf{x}) - \min(\mathbf{x}) \\ \text{s.t.} \quad & \mathcal{FB}(\mathbf{x}) \text{ or } \mathcal{SB}(\mathbf{x}) \text{ with } x_n \geq 0, \forall n. \end{aligned} \quad (19)$$

On the other hand, for  $\mathbf{x} \in \mathbb{C}^N$  (excitations with amplitude and phase), the problem is harder to solve and the formulation via the matrix  $\mathbf{X} = \mathbf{x}\mathbf{x}^H$  is required to keep the problem convex. A reduced DRR can be achieved by minimizing  $|\max_n |x_n|^2 - \min_n |x_n|^2|$ . The array synthesis problem is then:

$$\begin{aligned} \min_{\mathbf{X}} \quad & \left[ \max_n (\mathbf{X}_{n,n}) - \min_n (\mathbf{X}_{n,n}) \right] + \gamma \text{Tr}(\mathbf{X}) \\ \text{s.t.} \quad & \mathcal{FB}(\mathbf{X}) \text{ or } \mathcal{SB}(\mathbf{X}) \text{ and } \mathbf{X} \succeq 0. \end{aligned} \quad (20)$$

### D. Excitations with Smooth Variation

The array pattern synthesis with smoothly varying excitations can be achieved, when the excitations are real values, by minimizing the TV-norm as follows:

$$\begin{aligned} \min_{\mathbf{x} \in \mathbb{R}^N} \quad & \|\mathbf{x}\|_{\text{TV}} \\ \text{s.t.} \quad & \mathcal{FB}(\mathbf{x}) \text{ or } \mathcal{SB}(\mathbf{x}). \end{aligned} \quad (21)$$

When the excitations are expected to be complex, the smoothness can be synthesized by minimizing  $||x_{n+1}|^2 - |x_n|^2|$ . The problem is:

$$\begin{aligned} \min_{\mathbf{X}} \quad & \sum_{n=1}^N |\mathbf{X}_{n+1,n+1} - \mathbf{X}_{n,n}| + \gamma \text{Tr}(\mathbf{X}) \\ \text{s.t.} \quad & \mathcal{FB}(\mathbf{X}) \text{ or } \mathcal{SB}(\mathbf{X}) \text{ and } \mathbf{X} \succeq 0. \end{aligned} \quad (22)$$

## IV. NUMERICAL APPLICATIONS

Various array synthesis problems are presented to show the interest and efficiency of the proposed approaches. Generally speaking, better radiation performances could be achieved, e.g. lower sidelobe levels or smaller shaped beam ripples, but the goal here is to let some freedom in order to illustrate the effects of the proposed excitation regularizations.

### A. Sparse Array with Uniform Amplitude and Phase Excitations

The goal is to synthesize an array radiating a focused beam with excitations having all the same amplitude. For that purpose, the pattern radiated by the symmetric isophoric array obtained via a deterministic approach described in [3] is taken as the desired pattern  $y^d$ . The reweighted minimization problem (17) is applied and at each iteration the problem of the form of (16) is solved:

$$\min_{\mathbf{x}} \|\mathbf{x}\|_1 + \gamma \|\mathbf{x} - 0.5\|_{\infty} \quad \text{s.t.} \quad |\mathbf{a}(\theta)^H \mathbf{x} - y^d(\theta)| \leq \epsilon, \forall \theta$$

As detailed in Section III-A, we are looking for binary excitations ( $x_n \in \{0, 1\}$ ) which means that the sought-for

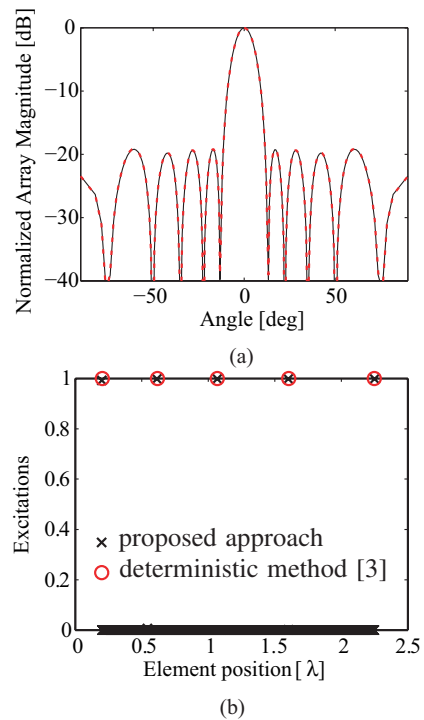


Fig. 2. Synthesis of an isophoric array by fitting the focused beam pattern of [3]. (a) Fitted and reconstructed radiation patterns, they are superimposed. (b) Synthesized excitations: the optimal excitations (red circles) of [3] are retrieved by the proposed procedure (black crosses), all excitations (except 5 of them) are set to zero.

$x_n$  are of equal distance to 0.5, hence the minimization of  $\|\mathbf{x} - 0.5\|_{\infty}$ . The upper bound  $\epsilon$  has to be small enough, here smaller than  $10^{-5}$ , in order to ensure a good fit between synthesized and desired pattern. Moreover, the parameter  $\gamma$  is chosen greater than  $2N$ , i.e. 10. If  $\gamma < 10$ , the radiation pattern is still well reconstructed but the excitations starts to have various magnitudes because more importance is then attributed to the  $\ell_1$  penalty than the  $\ell_{\infty}$  one.

At iteration  $k = 0$ , we start with a fully populated symmetric array composed of isotropic elements placed every  $0.001 \lambda$ . After five iterations of the reweighted minimization problem (17), five excitations only are remaining and we obtain the results shown in Fig. 2. This synthesis problem is solved using CVX [15] in less than 10 seconds on a standard laptop. The corresponding MatLab code is given in Appendix I.

The radiation patterns are superimposed and the excitations found by the proposed approach are very closed to those obtained in [3]. The element locations are not exactly the optimal ones computed by [3] because of the initial discretization of  $0.001 \lambda$  imposed by the thinning strategy.

### B. Arrays with Groups of Uniform Amplitude, Minimized DRR and ‘smooth’ Excitations

The goal is to synthesize a focused beam, of beamwidth  $\Delta\theta = 10^\circ$  and sidelobes below  $\rho = -15$  dB. We consider a linear array composed of 30 patches working at 10 GHz and uniformly spaced by a distance of  $0.6\lambda$ . These patches are simulated in their environment with the 3D full wave

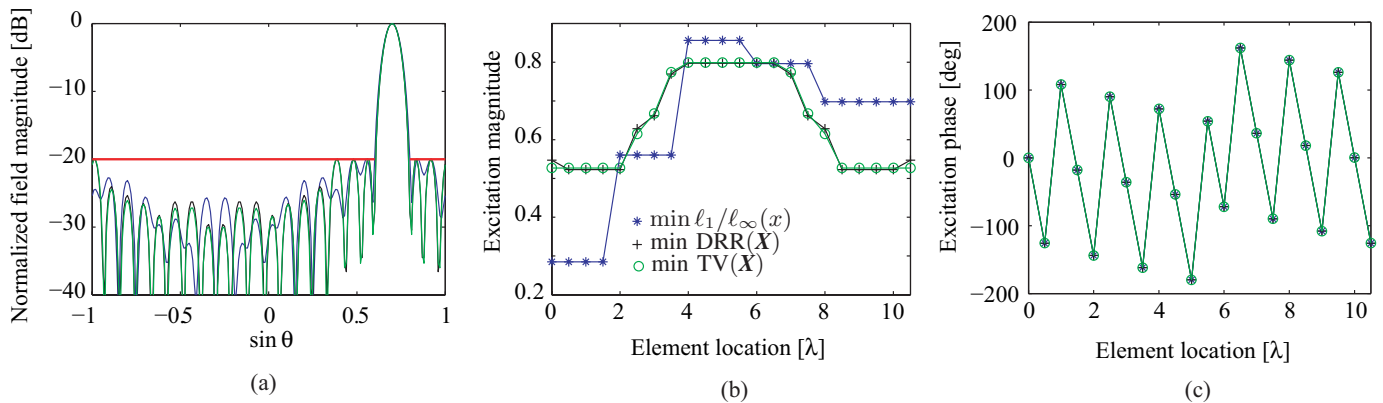


Fig. 4. Synthesis of linear array composed of 22 half wavelength spaced elements with 3 excitation objectives: 5 groups with uniform amplitude excitations, minimized DRR and smooth excitations. Synthesis results: (a) far field patterns, optimized excitation (b) magnitudes and (c) phases.

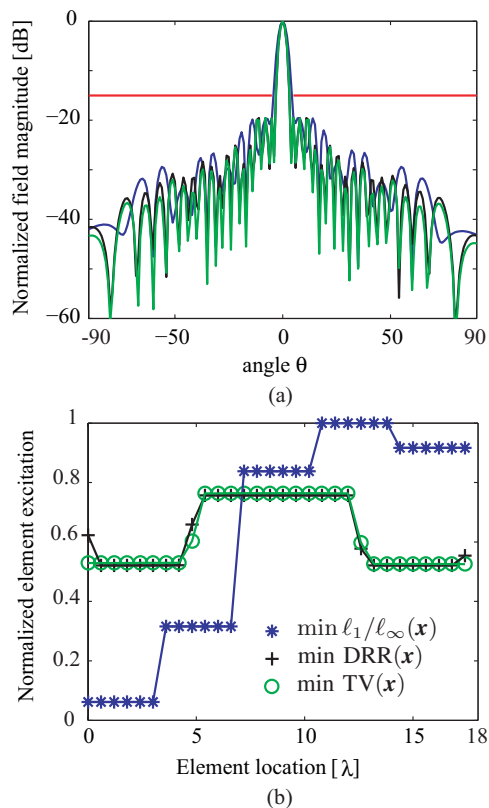


Fig. 3. Synthesis of a linear array composed of 30 patches working at 10 GHz and simulated using Ansys HFSS in order to fully take into account the coupling effects. Three excitation objectives are set (5 groups with uniform magnitudes, a minimized DRR and smooth excitations): (a) far field patterns and (b) optimized excitations.

electromagnetic software Ansys HFSS in order to build the steering vector  $\mathbf{a}(\theta)$  in (1). By doing so, the coupling effects between patches are fully taken into account.

Three types of regularization are used to synthesize the pattern radiated by the array while controlling its excitations in order to have 5 groups of uniform amplitude excitations, a minimized DRR and smooth excitations. Note that the choice of the number of groups is arbitrary. Of course, a

smaller number of groups reduces the degrees of freedom and consequently the array radiation performances. For that purpose, the convex optimization problems (18), (19) and (21) are used respectively.

The synthesized patterns, plotted in Fig. 3(a) show that the radiation requirements are fulfilled. The optimized excitations are reported in Fig. 3(b), the 5 groups of identical excitations are clearly visible. The minimization of the TV norm of  $\mathbf{x}$  leads to a distribution of the excitations with three levels. This distribution is close to the one with the smallest DRR represented by the black markers.

Let us now consider a linear array composed of 22 half wavelength spaced isotropic sources to radiate a focused beam in the direction  $\sin \theta_0 = 0.7$ . The beamwidth is defined by  $0.6 \leq \sin(\theta) \leq 0.8$  and the sidelobes are kept below -20 dB. Three excitation objectives are compared: 5 groups with uniform amplitudes, the minimization of the DRR and the smoothness of the excitations. For these two latter cases, the optimization problems (20) and (22) are solved since complex excitations are expected to radiate the off-axis beam.

The synthesis results are plotted in Fig. 4. The three synthesized patterns respect the predefined radiation constraints. The excitations have the same expected linear phase variation in order to radiate an off-centered beam. The excitation magnitude plots clearly show 5 groups of excitations having the same magnitude for the  $\ell_1/\ell_\infty$  minimization. The minimization of the DRR and the TV norm leads to very close results, the dynamic range of the excitation magnitudes and the variation of amplitude between two excitations is obviously much smaller than the case of the ‘amplitude-group’ optimization. Finally, the synthesis of a cosecant beam with a linear array composed of 21 half wavelength spaced elements is addressed. The goal is to fit the shaped beam pattern provided by [27] (constraints of the form (10)) while controlling the excitations of the array to synthesize an array with: either 7 groups of uniform magnitude, a minimized DRR or smoothly varying excitations.

The synthesized shaped beams are plotted in Fig. 5(a), they all fit closely the desired cosecant pattern. The three sets of optimized excitations are given in Fig. 5(b), their own

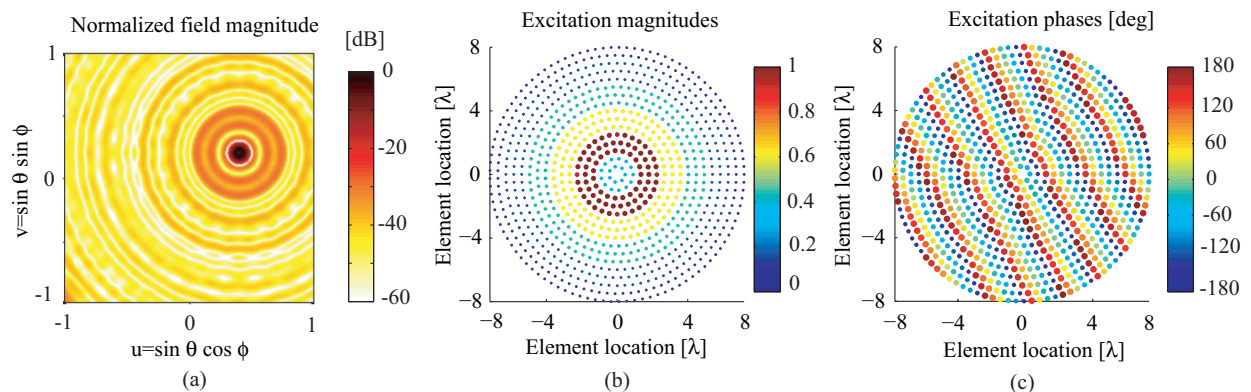


Fig. 6. Synthesis of a circular array composed of 862 elements with 6 zones of uniform excitation magnitudes ( $\ell_1/\ell_\infty$  norm minimization): (a) far field pattern, optimized excitation (b) magnitudes (6 zones are clearly visible) and (c) phases.

characteristics are clearly noticeable.

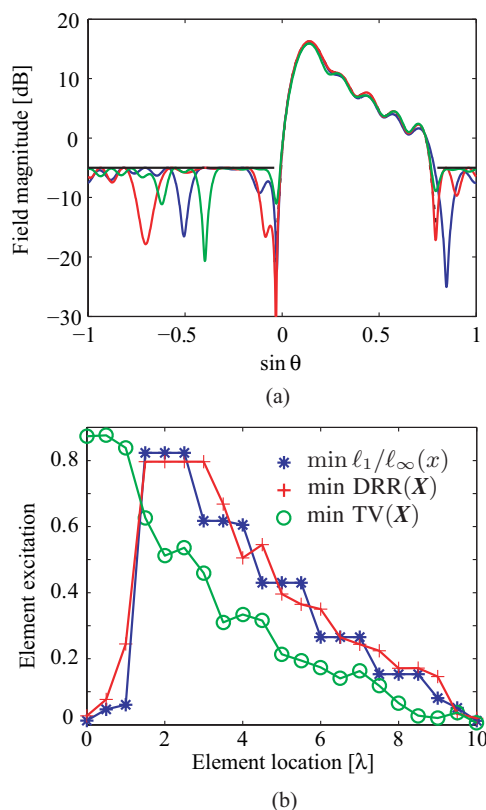


Fig. 5. Synthesis of a cosecant pattern by fitting the shaped beam provided in [27] with a linear array composed of 21 half wavelength spaced elements. Three excitation objectives are considered : 7 groups with uniform amplitude excitations, minimized DRR and smooth excitations. Synthesis results: (a) far field patterns and (b) optimized excitation magnitudes.

### C. Planar Array with Groups of Equi-Amplitude Excitations

Let us consider a planar array composed of 17 rings for a total of 862 isotropic elements, the layout is represented in Fig. 6(b) and (c). The goal is to synthesize a focused beam tilted in the direction ( $u_0 = 0.4, v_0 = 0.2$ ) where the main beam is defined by the region such as  $\sqrt{(u - u_0)^2 + (v - v_0)^2} \leq 0.1$

while the sidelobes are kept below  $-30$  dB. The 17 rings of the array are divided into 6 zones (of 3 rings except for the inner one) on which a uniform excitation magnitude is imposed so as to ease the excitation of such array. The mixed  $\ell_1/\ell_\infty$  norm minimization (18) is used to synthesize this array. The results are obtained in less than 30s with a standard laptop and they are reported in Fig. 6. The uniform excitation amplitude zones are clearly visible in Fig. 6(b) and the synthesized phase gradient required to tilt the beam is plotted in Fig. 6(c). Note that the excitation magnitudes of the 6 zones have been optimized as well as the phases of each element.

## V. CONCLUSION

A collection of regularization schemes has been presented to synthesize antenna array patterns while controlling the element excitations. By minimizing the appropriate norm (mixed  $\ell_1/\ell_\infty$ , total variation or a combination of the  $\ell_1$  and  $\ell_\infty$ ) of the array pattern solution, it is possible to synthesize either uniform amplitude sparse arrays, arrays with a smooth excitation amplitude variation or arrays with a reduced dynamic range ratio. All proposed approaches can take into account the coupling between the array antennas. Moreover, they are easy to implement and use only off-the-shelf routines.

Various numerical application examples have been considered to illustrate the potentialities of the proposed approaches. Although the formulations are all convex, the optimality of the solution can, in general, not be guaranteed because of necessary approximations. However, our approach is able to retrieve the uniform amplitude sparse array synthesized by a deterministic technique.

The proposed techniques could also be of great relevance for the synthesis of transmit-arrays and reflectarrays where, in general, only the phases of the elements are optimized and/or smooth magnitude variations are desired.

## APPENDIX I

### A CVX CODE FOR UNIFORM AMPLITUDE ARRAYS

% Inputs

% N: length of the unknown vector x

% yd: desired far field pattern

```

% A: array factor
% IT_nb: number of iterations

W = eye(N); % reweighting matrix

for k=1:IT_nb
    cvx_begin
        variable x(N)
        minimize(norm(W*x,1) + gamma*norm(x-alpha,inf))
        subject to
            norm(A*x-yd ,2) <= epsilon;
    cvx_end
    W = diag(1./(abs(x)+delta));
end

% Settings for example IV-A

gamma = 10;
epsilon = 1e-5;
delta = 1e-2;
alpha = 0.5;
IT_nb = 5;

```

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**Benjamin Fuchs** (S'06-M'08-SM'16) received the M.S. and electrical engineering degrees in 2004 from the National Institute of Applied Science of Rennes, France. He received the Ph.D. degree in 2007 from the University of Rennes 1, France, and was during that period a visiting scholar at the University of Colorado at Boulder, USA. In 2009, he joined the Institute of Electronics and Telecommunications of Rennes (IETR) as a researcher at the Centre National de la Recherche Scientifique (CNRS). He has spent three years (2008 as postdoctoral research fellow and 2011-2012 on leave from CNRS) at the Swiss Federal Institute of Technology of Lausanne (EPFL) in Switzerland. His research interests revolve around the analysis and synthesis of electromagnetic field for antenna design and microwave imaging.





**Sébastien Rondineau** was born in 1975 in France. He earned his *Diplôme d'Ingénieur en Informatique et téléCommunications*, a postgraduate degree in signal processing and telecommunications, in 1999 and his Ph.D. degree in 2002 from the University of Rennes 1, France. He was Research Associate (2002-2006) and Assistant Research Professor (2006-2008) at the Microwave and Active Antenna Laboratory of the Electrical and Computer Engineering Department, University of Colorado at Boulder, USA. From 2008 to 2015, he was working in various

RF/Electronic Warfare Brazilian companies as technical director and/or company founder. Since February 2015, he is now Professor at both the Aerospace Engineering and the Electronic Engineering departments of the University of Brasilia, Brazil. His research interests include beam-forming, RF-optical conversion, computational electromagnetics, mode matching, conformal mapping, propagation and scattering of waves, dielectric lenses, discrete lens arrays and antennas.