When the warping stiffness increases, the analytical model hardly overestimates (of no more than 5%) the value of the elastic critical bending moment when compared to finite element analyses. When warping is fixed, the analytical model becomes slightly safer, with a difference below 3%.

When ψ = -1, the results given by the different methods coincide up to a higher warping restraint. When the warping stiffness increases, the analytical model lays on the safe side but the difference with the shell model is not greater than 10%.

The variation between Piotrowski [6] and the analytical model is only visible when the warping restraints are significant, particularly when ψ = -1. When compared to finite element analyses using shell elements, the analytical model always lays on the safe side unlike Piotrowski [6].

Table 6 shows statistical parameters about the calculated values of the critical moment of the beams P1 to P5 (included) presented in Table 5. The ratios between critical bending moments determined with the analytical model and the shell finite element model are presented in Table 6. This table also shows the relations between the critical bending moment increases $M_{cr}/M_{cr,0}$ calculated with the analytical model and FE analyses.

Table 6 Comparison between analytical results and shell finite element analyses

ψ		1	0,5	0	-0,5	-1
$\dfrac{\left(\frac{M_{cr}}{M_{cr,0}}\right)_{an}}{\left(\frac{M_{cr}}{M_{cr,0}}\right)_{FEA}}$	*M*	*1,006*	*1,006*	*1,007*	*1,005*	*0,976*
	SD	0,0201	0,0201	0,0206	0,0178	0,0352
$\dfrac{M_{cr,an}}{M_{cr,FEA}}$	*M*	*1,017*	*1,021*	*1,043*	*1,069*	*0,954*
	SD.	0,0207	0,0206	0,0220	0,0275	0,0345

Where:

- M: Mean
- SD: Standard deviation

The elastic critical bending moment evolution estimated by the analytical model is very close to the results of shell finite element analyses. The difference is lower than 1% except when ψ = -1 where it increases up to 2,4%.

Critical bending moments determined with the analytical model and FEA are slightly different, but their difference is less than 7%, particularly when ψ is between -0,5 and 0. It matches with the values of ψ for which the analytical and numerical models show small differences in the value of C_1 for free warping (see Figure 8).

The displacement and rotation fields could still be refined with a third term to improve results. However, it would lead to heavier analytical developments, leading to a very cumbersome expression for the equivalent uniform moment factor C_1. In addition, 7% is still an admissible difference given that the resistant bending moment determined according to Eurocode 3 Part 1-1 [7], $M_{b,Rd}$, depends not only on the critical bending moment but also on many other parameters.

5 Conclusions

The energy method has been used to derive an analytical expression of the elastic critical bending moment of a beam with uniform doubly symmetric I-section and warping restraints at both ends, subjected to a uniform or linear bending moment distribution. The analytical expression is the same as the French National Annex to Eurocode 3 Part 1-1 [2], but the warping coefficient k_w and the equivalent uniform moment factor C_1 have distinct expressions.

The warping coefficient k_w has been shown to remain unchanged whatever the bending moment distribution and only depends upon the warping stiffness of the beam EI_w, its length L and the stiffness of the warping restraints c_w. Furthermore, the equivalent uniform moment factor C_1, only based on the bending moment distribution in the French NA [2], is found to also be depending on the stiffness c_w of the warping restraints.

Results from the analytical model were shown to be in good agreement with finite element analyses computed with the software packages ANSYS and LTBeamN [1].

The increase in the value of the critical moment is limited if the warping restraints are only obtained by end plates, but the benefit can become significant if different end connections are considered (fixed column bases, beam-to-beam or beam-to-column joints…).

Future work may include the derivation of analytical expressions to determine the warping stiffness corresponding to realistic support conditions. In addition, the bending moment distribution may be extended to generalise the expression of the elastic critical moment.

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